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Sticky Behavior of Fluid Particles in the Compressible Kraichnan Model

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We consider the compressible Kraichnan model of turbulent advection with small molecular diffusivity and velocity field regularized at short scales to mimic the effects of viscosity. As noted in ref. 5, removing those two regularizations in two opposite orders for intermediate values of compressibility gives Lagrangian flows with quite different properties. Removing the viscous regularization before diffusivity leads to the explosive separation of trajectories of fluid particles whereas turning the regularizations off in the opposite order results in coalescence of Lagrangian trajectories. In the present paper we re-examine the situation first addressed in ref. 6 in which the Prandtl number is varied when the regularizations are removed. We show that an appropriate fine-tuning leads to a sticky behavior of trajectories which hit each other on and off spending a positive amount of time together. We examine the effect of such a trajectory behavior on the passive transport showing that it induces anomalous scaling of the stationary 2-point structure function of an advected tracer and influences the rate of condensation of tracer energy in the zero wavenumber mode.

KEY WORDS: Kraichan model; Lagrangian flows.

1. INTRODUCTION

Certain qualitative features of transport of scalar or vector quantities by turbulent flows may be understood by ignoring the back-reaction of the transported quantity on the velocity dynamics (passive advection). In such situations the transport properties reflect closely the properties of the fluid trajectories, i.e. of the Lagrangian flow. The effects of molecular diffusion

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may be also taken into account by simply perturbing the fluid trajectories by a Brownian noise. In fully developed turbulence the Lagrangian flow exhibits different regimes. In particular, the pair dispersion (the distance between two trajectories) will grow like square root of time when dominated by the molecular noise, like exponential of time times the Lyapunov exponent at short distances below the Kolmogorov viscous scale if the diffusive effects may be neglected and superdiffusively (like square root of time cubed in the celebrated Richardson law[21]) in the inertial interval of scales. Even if not clearly separated, these regimes influence advection in a different way and the transport properties become their cumulative effect. In particular, in the presence of compressibility the (top) Lyapunov exponent in the viscous interval might become negative so that diffusivity and viscosity would have opposite effect on the Lagrangian dispersion.

In this paper we shall examine such a situation in the Kraichnan model of turbulent flow[15] where the realistic velocity ensemble is replaced by a Gaussian one with temporal decorrelation. Although grossly oversimplified, the model admits different regimes and, due to its simplicity, analytic calculations. It has become a popular test-ground for the turbulence theory.[7] In ref.,[6] E and Vanden-Eijnden have argued that in the Kraichnan model with intermediate amount of compressibility it is possible to fine tune the divergence of the Prandtl number $Pr=\frac{\nu}{\kappa}$ when diffusivity κ and viscosity ν are sent to zero in such a way that the mean time that two noisy Lagrangian trajectories starting from the same point spend within the (shrinking) viscous scale has a finite limit. The fine tuning of Pr is designed to balance the diffusive spread of the pair on the shortest scales by the compression in the viscous interval. Although the effect was clearly identified in,[6] the required behavior of the Prandtl number was incorrectly characterized there. In the present paper, we re-examine a similar limit both from the point of view of the stochastic diffusion process describing the particle dispersion and of its generator. We show that under the properly identified fine tuning procedure for Pr, one obtains when κ and ν tend to zero the so called sticky diffusion process for pair dispersion. The generator of the sticky diffusion corresponds to a boundary condition that involves two derivatives at zero dispersion. We examine the consequences of such a limiting behavior for the transport properties of passively advected tracer. A more detailed discussion of the content and the results of the paper is deferred to Subsection 2.6.

2. LAGRANGIAN FLOW IN KRAICHNAN VELOCITIES

The transport in a synthetic Gaussian ensemble of time-decorrelated velocities was first considered by Kraichnan[15] and Kazantzev[14] for passively advected tracer and magnetic field, respectively. In the last decade, the model has led to a new understanding of the statistical intermittency of advected quantities and of the interplay between fluid compressibility and transport properties, see ref. 7 and references therein. As already mentioned, the passive advection by a d-dimensional velocity field $\vec{v}(t,\vec{r})$ is intimately related to the behavior of the Lagrangian trajectories (i.e. trajectories of fluid particles) perturbed by noise. Such trajectories satisfy the stochastic ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{R}(t)=\vec{v}(t,\,\vec{R}(t))+\sqrt{2\kappa}\,\vec{\eta}(t), \tag{1}$$

where $\vec{\eta}(t)$ is a d-dimensional white noise independent of velocity (and of noises of other particles) and $\kappa>0$ is the (molecular) diffusivity. The advection of a passive tracer $\theta(t,\vec{r})$ is governed by the advection-diffusion equation

$$\partial_t\theta+\vec{v}\cdot\vec{\nabla}\theta-\kappa\vec{\nabla}^2\theta=f\,, \tag{2}$$

where $f(t,\vec{r})$ is a source or forcing term. In the absence of forcing (i.e. for $f=0$) the passive tracer is carried along the particle trajectories so that

$$\theta(t,\vec{r})=\overline{\theta(0,\,\vec{R}(0;t,\vec{r}))}\,. \tag{3}$$

Here $\vec{R}(t';t,\vec{r})$ denotes a particle trajectory that passes through point $\vec{r}$ at time t and the overline stands for the average with respect to the noise $\vec{\eta}(t)$. Note that the forward tracer evolution is described by Lagrangian trajectories going backwards in time. Similarly, in the presence of the source, the tracer evolves according to the equation

$$\theta(t,\vec{r})=\overline{\theta(0,\,\vec{R}(0;t,\vec{r}))}+\int_0^t\overline{f(s,\,\vec{R}(s;t,\vec{r}))}\,\mathrm{d}s\,. \tag{4}$$

This way, in random velocities, the statistics of advected quantities like θ is linked to the statistical properties of (noisy) trajectories. The present article exploits this relation in yet another situation in the Kraichnan model of turbulent advection.

2.1. Kraichnan Ensemble

The Kraichnan model describes advection by a stochastic velocity field $\vec{v}(t,\vec{r})$ with Gaussian mean-zero statistics, stationary and decorrelated in time, homogeneous and isotropic in space and with fixed compressibility degree $\wp$ characterizing the relative strength of the incompressible and potential components of the velocity. As in any centered Gaussian ensemble, the statistics of velocities is totally characterized by the 2-point function $\langle\vec{v}(t,\vec{r})\otimes\vec{v}(t',\vec{r}\,')\rangle$. In what follows, we shall need the statistics of the equal-time velocity differences that is determined by the reduced expression

$$\frac{1}{2}\,\langle(\vec{v}(t,\vec{r})-\vec{v}(t',\vec{r}\,'))^{\otimes 2}\rangle=\delta(t-t')\,D_0\int\frac{1-e^{i\vec{k}.(\vec{r}-\vec{r}\,')}}{|\vec{k}|^{d+\xi}}\,\vec{\vec{P}}(\vec{k},\wp)\,\frac{d\vec{k}}{(2\pi)^d}$$
$$\equiv\delta(t-t')\,\vec{\vec{d}}(\vec{r}-\vec{r}\,'). \qquad (5)$$

The rank 2 tensor $\vec{\vec{d}}(\vec{r}\,'-\vec{r})$ is the second order spatial structure function of the velocity field. The rank 2 tensor $\vec{\vec{P}}(\vec{k},\wp)$, invariant under rotations and of trace 1, is taken to be

$$P_{ij}(\vec{k},\wp)=\frac{1-\wp}{d-1}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)+\wp\frac{k_ik_j}{k^2}$$

with $0\leq\wp\leq 1$. The coefficient $\wp$ is equal to $\frac{\partial_i\partial_j d_{ij}(\vec{r})}{\partial_i\partial_i d_{jj}(\vec{r})}$ (summation convention!) for all $\vec{r}\neq 0$, so that it is meaningful to characterize $\wp$ as the ratio $\frac{\langle(\partial_i v_i)^2\rangle}{\langle(\partial_i v_j)^2\rangle}$ even though the velocity field is not smooth for $\xi<2$. The value $\wp=0$ corresponds to an incompressible velocity field, whereas $\wp=1$ to a potential (e.g. irrotational) one. Note however that if the physical space is one dimensional (case $d=1$) then necessarily $\wp=1$ because a one-dimensional velocity field is always potential. One simply takes $\vec{\vec{P}}=1$ in this case. The spatial velocity structure function $\vec{\vec{d}}(\vec{r})$ scales in $|\vec{r}|$ with power ξ. We shall take ξ between 0 and 2. This choice guarantees that the typical velocity realizations are non-Lipschitz. More exactly, they are Hölder-continuous with exponent $\frac{1}{2}\xi'$ for any $\xi'<\xi$ (the realistic turbulent velocities are believed to have Hölder exponent $\leqslant\frac{1}{3}$ in the limit of infinite Reynolds numbers[19]). Note that the (positive) constant D_0 in (5) has the dimension $\frac{(\text{length})^{2-\xi}}{\text{time}}$.

2.2. Regularizing Effects

Real flows are always regularized at small scales by viscous effects. That is, the velocity field is smooth and the power law scaling with $\xi<2$ is observed only at distances much larger than the *viscous cutoff scale* l_ν

that becomes very small only for high Reynolds numbers (if the integral scale of turbulence is kept fixed). The small $\vec{r}$ behavior $\propto|\vec{r}|^\xi$ of $\vec{d}(\vec{r})$ above comes from the slow decay at large $\vec{k}$ of its Fourier transform (the term under the integral sign in (5)). We shall mimic the effect of viscosity in the present model by introducing an ultraviolet regulator and replacing $\vec{d}(\vec{r})$ by

$$\vec{d}(\vec{r};l_\nu) \equiv D_0 \int \frac{1-e^{i\vec{k}.\vec{r}}}{|\vec{k}|^{d+\xi}} \vec{P}(\vec{k},\wp)\, f(l_\nu|\vec{k}|)\frac{\mathrm{d}\vec{k}}{(2\pi)^d} \tag{6}$$

where the function f is taken positive, smooth, decreasing, fast decaying at infinity and with $f(0)=1$, $f'(0)=0$. The viscosity ν itself may be defined as

$$\nu = D_0 l_\nu^\xi \tag{7}$$

which is the only combination of D_0 and l_ν of dimension $\frac{(\mathrm{length})^2}{\mathrm{time}}$.

Another regularizing effect in real flows comes from finite molecular diffusivity. The tracer field $\theta(t,\vec{r})$ passively transported by the flow will be smoothed by diffusion, see (3). In typical velocities, this smoothing effect becomes important at scales much smaller than the *diffusive cutoff scale* l_κ. Beneath this scale diffusion dominates advection. We modeled molecular diffusivity by adding white noise terms to the fluid particle velocities (independent ones for different particles, see (1)). In the Kraichnan model, the scale l_κ beneath which diffusion dominates advection may be expressed in terms of κ, D_0 and l_ν, with $l_\kappa=0$ corresponding to $\kappa=0$. The expression may be inverted to calculate κ in terms of D_0, l_κ, l_ν. Later on we shall specify such a relation in the case when $l_\kappa \ll l_\nu$, see (28). This will provide an expression for the Prandtl number $Pr \equiv \frac{\nu}{\kappa}$.

2.3. Statistics of Fluid Particles

Even in a simple random velocity ensemble, the statistics of the Lagrangian flow may be quite complicated. It may be studied by looking at the joint N-particle probability density functions (PDFs) defined by

$$P_N^{t,t'}(\vec{r}_1,\ldots,\vec{r}_N;\vec{r}_1',\ldots,\vec{r}_N') = \left\langle \prod_{n=1}^{N} \overline{\delta(\vec{r}_n' - \vec{R}(t';t,\vec{r}_n))} \right\rangle. \tag{8}$$

Here, as before, the overline denotes the average over the (independent) white noises and $\langle\cdots\rangle$ stands for the velocity ensemble average. In the Kraichnan model, due to the temporal decorrelation of velocities, the

PDFs (8) are Markovian and they define a consistent hierarchy of N-particle stationary Markov processes that contains the complete information about the statistics of the Lagrangian flow and of the velocities themselves.[17] In this paper we shall be interested uniquely in the behavior of the separation of a pair of Lagrangian particles. The main object of our interest will be the PDF of finding their time t' separation equal to $\vec{r}'$, given that their time t separation is equal to $\vec{r}$:

$$P^{t,t'}(\vec{r},\vec{r}\,')=\left\langle\overline{\delta(\vec{r}\,'-\vec{R}(t';t,\vec{r})+\vec{R}(t';t,\vec{0}))}\right\rangle.$$

In stationary, time-reversal invariant velocity ensembles, such as the Kraichnan model, $P^{t,t'}$ depends only on $|t-t'|$ and we may use the notation $P^{|t-t'|}$.

The PDF $P^{t,t'}(\vec{r},\vec{r}\,')$ governs the free decay of the 2-point function of the passive tracer, evolving according to equation (2)[4] with $f=0$. If at time zero the tracer is distributed independently of the velocity field and the trajectory noises, with a homogeneous 2-point function $\langle\theta(0,\vec{r}_1)\,\theta(0,\vec{r}_2)\rangle=F(0,\vec{r}_2-\vec{r}_1)$, then at a later time t its 2-point function is given by

$$F(t,\vec{r})=\int F(0,\vec{r}\,')\,P^{t,0}(\vec{r},\vec{r}\,')\,\mathrm{d}\vec{r}\,' \tag{9}$$

as follows from (3). Similarly, if the scalar source f is a random field with mean zero and 2-point function

$$\langle f(t,\vec{r})\,f(t',\vec{r}\,')\rangle=\delta(t-t')\,\chi(\vec{r}-\vec{r}\,') \tag{10}$$

and it is independent of the velocity field, the trajectory noises and the initial scalar distribution, then the evolution of the tracer 2-point function is described by the relation

$$F(t,\vec{r})=\int F(0,\vec{r}\,')\,P^{t,0}(\vec{r},\vec{r}\,')\,\mathrm{d}\vec{r}\,'+\int_0^t \mathrm{d}s\int\chi(\vec{r}\,')\,P^{t,s}(\vec{r},\vec{r}\,')\,\mathrm{d}\vec{r}\,' \tag{11}$$

as follows from (4) by taking averages.

Our aim is to describe the particle pair separation in Kraichnan velocities at large scales, much larger than the cutoff scales l_ν and l_κ. The effective description will depend on the Prandtl number and on the scales

[4] In the Kraichnan velocity that is white in time, Eq. (2) should be interpreted as a Stratonovich stochastic differential equation, see e.g. Sect. II.C.3 of ref. 7.

involved. Instead of maintaining the cutoffs finite, we want to give an effective large scale theory without cutoffs but with some specific boundary condition at vanishing separation. Such a theory should give rise to the same behavior at the large scales of interest. An alternative way of doing this is by taking the cutoffs l_ν and l_κ to zero in a fashion that preserves large scale behavior and to examine the boundary condition that arises in this limiting process.

2.4. Lagrangian Dispersion

As first derived in ref. 15, the time evolution of the scalar 2-point function of the tracer in the Kraichnan ensemble of velocities is governed by the differential equation

$$\partial_t F = \mathcal{M}_{\nu,\kappa} F$$

where $\mathcal{M}_{\nu,\kappa}$ is a partial differential operator which can be written in terms of the velocity structure function $\vec{\vec{d}}$ and diffusivity κ as

$$\mathcal{M}_{\nu,\kappa} \equiv d_{ij}(\vec{r}; l_\nu)\partial_i \partial_j + 2\kappa \vec{\nabla}^2.$$

It follows that the separation PDF takes the heat kernel form

$$P^t(\vec{r}_0, \vec{r}) = e^{\mathcal{M}_{\nu,\kappa}}(\vec{r}_0, \vec{r}). \tag{12}$$

As we shall be only interested in distances between two particles, usually called the *Lagrangian pair dispersion*, and not in the angular distribution of the particle separation, we may project (12) to the rotationally invariant sector. This is done by restricting the action of operator $\mathcal{M}_{\nu,\kappa}$ to functions of the radial variable $r = |\vec{r}|$ only. $\mathcal{M}_{\nu,\kappa}$ is a rotationally invariant operator and it maps functions of r to functions of r. We shall denote by $M_{\nu,\kappa}$ its restriction to such functions. It is easy to show that

$$M_{\nu,\kappa} = \tilde{D}_0 \left[f_1(r)\partial_r^2 + f_2(r)\partial_r \right] \tag{13}$$

with the coefficient functions $f_{1,2}(r) \equiv f_{1,2}(r; l_\nu, l_\kappa)$ expressed in terms of the spatial velocity structure function $\vec{\vec{d}}$ by

$$\tilde{D}_0 f_1(r) = d_{ij}(\vec{r}; l_\nu)\frac{r_i r_j}{r^2} + 2\kappa \tag{14}$$

$$\tilde{D}_0 f_2(r) = \frac{1}{r} d_{ij}(\vec{r}; l_\nu)(\delta_{ij} - \frac{r_i r_j}{r^2}) + 2\frac{d-1}{r}\kappa \tag{15}$$

for any $\vec{r}$ such that $|\vec{r}|=r$. For detailed calculation of the values of f_1 and f_2 see Appendix F. It turns out that, for r much larger than the cut-off scales l_κ, l_ν, the function f_1 is asymptotically proportional to r^ξ and the function f_2 to $r^{\xi-1}$. We choose the constant $\tilde{D}_0$ proportional to D_0, see (5), so that $r^{-\xi} f_1(r) \to 1$ when $r \to \infty$.

In order to turn $M_{\nu,\kappa}$ into a true operator on the half-line $\mathbb{R}_+$, we also have to specify its domain $\mathcal{D}(M_{\nu,\kappa})$. This boils down to the choice of the boundary condition to impose at zero. The domain of $\mathcal{M}_{\nu,\kappa}$, as long as κ is positive, is some locally H^2 Sobolev space. For its rotationally invariant elements this implies that their gradient vanishes at $\vec{r}=0$ resulting in the Neumann boundary condition $\partial_r|_{r=0}=0$ for $M_{\nu,\kappa}$. This way equation (12) reduces in the rotationally invariant sector to the heat kernel expression for the particle distance PDF

$$P^t(r_0, r) = e^{tM_{\nu,\kappa}}(r_0, r), \tag{16}$$

where $M_{\nu,\kappa}$ is the second order differential operator on the half-line of $r \geqslant 0$ given by (13), taken with the Neumann boundary condition at $r = 0$. We may view the right hand side of (16) as the transition probability density of a Markov process $r(t)$ that describes the distance between two noisy Lagrangian particles in the random flow and treat the underlying diffusion process with the adapted tools, see Appendix A. Alternatively, we may interpret (16) as describing kernels of a semigroup of operators and apply to it the usual analysis toolkit. Both methods will be developed.

2.5. Initial Classification of Boundary Behaviors

The possible boundary behaviors at the origin of the one-dimensional diffusion process $r(t)$ describing the inter-particle distance in the absence of regularizations (i.e. for $l_\nu, l_\kappa = 0$) may be easily classified.[6] In this case the generator (13) of the process reduces to the operator

$$M \equiv M_{0,0} = \tilde{D}_0 r^\xi \left(\partial_r^2 + \frac{a_{\xi,\wp}}{r}\partial_r\right) \tag{17}$$

with

$$a_{\xi,\wp} = \frac{d+\xi}{1+\xi\wp} - 1\,, \tag{18}$$

see Appendix F. Let us note in passing for later use that we have the implication

$$a_{\xi,\wp} < 1 \quad \Longrightarrow \quad a_{2,\wp} < 1\,.$$

For $d \geqslant 2$ this is a simple consequence of (18). In the case $d=1$ recall that $\wp=1$ and then $a_{\xi,1}=0$ for any value of ξ. Below, whenever the value of the compressibility degree $\wp$ is understood, we shall write a_ξ instead of the more unwieldy $a_{\xi,\wp}$ and sometimes simply a if no confusion may arise.

In the case $\xi<2$ corresponding to spatially rough flows, we can introduce the new coordinate

$$u=u(r)\equiv\tfrac{2}{2-\xi}\,\tilde{D}_0^{-1/2}r^{1-\frac{\xi}{2}}\,. \tag{19}$$

In this coordinate the generator (17) becomes

$$\partial_u^2+\left(\frac{2a_\xi-\xi}{2-\xi}\right)\frac{1}{u}\partial_u \tag{20}$$

and it describes the Bessel process (see[2] for a brief *résumé*) of *parameter* $-b_\xi$ or *effective dimension* $d_{\rm eff}=2(1-b_\xi)$ where

$$b_\xi=\frac{1-a_\xi}{2-\xi}\,. \tag{21}$$

If $d_{\rm eff}$ is a positive integer then the corresponding Bessel process describes the behavior of the norm $|\vec{W}(t)|$ of the $d_{\rm eff}$-dimensional Brownian motion $\vec{W}(t)$.

The general theory of the boundary behaviors of one-dimensional diffusion processes was laid down by Feller in ref. 8. For the Bessel process, the boundary behavior at zero is well known[2,22] and it depends on the parameter or equivalently the effective dimension of the process. For $d_{\rm eff}\le 0$, zero is an exit boundary point (the realizations may arrive at zero in a finite time, but they cannot come back from zero nor start there). For $0<d_{\rm eff}<2$, zero is a regular boundary point (the realizations can go to and leave zero in finite time). For $d_{\rm eff}\geqslant 2$, zero is an entrance boundary (the realizations may start from zero, but no realization goes to zero in finite time). The different boundary behaviors enumerated above correspond respectively to the regimes in the Kraichnan model that were coined *strongly compressible* ($\wp\geqslant\frac{d}{\xi^2}$), *of intermediate compressibility* ($\frac{d-2}{2\xi}+\frac{1}{2}<\wp<\frac{d}{\xi^2}$) and *weakly compressible* ($\wp\le\frac{d-2}{2\xi}+\frac{1}{2}$), see.[11,5]

In the limiting case $\xi=2$ corresponding to spatially smooth flows, we may introduce the new coordinate $u=\tilde{D}_0^{-1/2}\ln r$. In this coordinate the generator (17) becomes

$$\partial_u^2+(a_2-1)\partial_u$$

Table I. Regimes of the Unregularized Kraichnan Model

Flow	Compressibility	Characterization	Implications	Boundary
rough	weak	$\wp \leqslant \frac{d-2}{2\xi} + \frac{1}{2}$	$a_{\xi,\wp} \geqslant 1$	entrance
	intermediate	$\frac{d-2}{2\xi} + \frac{1}{2} < \wp < \frac{d}{\xi^2}$	$1 > a_{\xi,\wp} > \xi - 1$ $1 > a_{2,\wp}$	regular
	strong	$\wp \geqslant \frac{d}{\xi^2}$	$\xi - 1 \geqslant a_{\xi,\wp}$	exit
smooth	weak	$\wp \leqslant \frac{d}{4}$	$a_{2,\wp} \geqslant 1$	natural
	strong	$\wp \geqslant \frac{d}{4}$	$a_{2,\wp} \leqslant 1$	

and it describes the one-dimensional Brownian motion viewed in the frame moving with speed $-(a_2 - 1)$ (i.e. the Brownian motion with a drift). In this case $r = 0$ (corresponding to $u = -\infty$) is a natural boundary point with no realizations that attain it or leave it in finite time. The quantity $(a_2 - 1)\tilde{D}_0^{1/2} = \frac{d-4\wp}{1+2\wp}\tilde{D}_0^{1/2}$ is the Lyapunov exponent of the Lagrangian flow and it is non-negative in the weakly compressible regime $\wp \leq \frac{d}{4}$, and non-positive in the strongly compressible one $\wp \geqslant \frac{d}{4}$, vanishing at their common point $\wp = \frac{d}{4}$. We sum up this classification in Table I.

The first three lines correspond to rough velocity fields with $\xi < 2$. The last two correspond to smooth ones with $\xi = 2$. Note that for $d \geqslant 4$ the weakly compressible regime extends to the whole interval $0 \leq \wp \leq 1$.

The relationship between the boundary behavior of the one-dimensional diffusion process $r(t)$ describing the distance between Lagrangian trajectories, on the one hand, and the different regimes of the Lagrangian flow, on the other hand, may be explained intuitively. If the realizations can go to zero but cannot come back, then they are trapped there. This implies coalescence of Lagrangian trajectories that characterizes the strong compressibility regime, as first observed in ref. 11. If realizations can go to and come back from zero, then one has to specify further the boundary behavior. There may be (and, as we find, there are indeed) different possible behaviors of Lagrangian trajectories when they meet. This is what happens for intermediate compressibility, as first noted in ref. 5. If realizations starting from inside the half-line can never reach zero then no trapping is possible. In this situation, corresponding to weak compressibility, the realizations may, however, enter from zero if $\xi < 2$ meaning that the Lagrangian trajectories separate explosively leading to a non-deterministic Lagrangian flow.[1] For $\xi = 2$ the realizations can neither collapse to zero nor explode from it. In smooth flows, different trajectories remain separated indefinitely in the future and the past, although they may behave chaotically, with small separations growing exponentially in time.

2.6. Main Results

In any case, the regulated generator $M_{\nu,\kappa}$ of Eq. (13) taken with Neumann boundary condition at zero gives rise to a diffusion process $r(t)$ on the half-line describing the statistics of the Lagrangian dispersion. We would like to know how this process behaves in the limit when $l_\nu, l_\kappa \to 0$.

The cases of the weakly and strongly compressible regimes where the limiting dispersion process does not depend on the way in which the limit is taken seem to have been treated adequately.[11,5] In these cases only one process is possible in the limit and it corresponds either to instantaneous reflection of trajectories when they meet (for weak compressibility) or to coalescence of trajectories upon the first hit (for strong compressibility).

In ref. 5,6 it was argued that, in the intermediate regime, the possible limiting processes are described by the unregulated generators M of Eq. (17) with boundary conditions at zero depending on the way the limit $l_\nu, l_\kappa \to 0$ is taken. Our analysis following similar lines (estimation of mean time spent by two trajectories at distances $\leq l_\nu$) confirms this picture but differs from ref. 6 in specific conclusions. The reason for the discrepancy is the wrong estimate (56) in ref. 6 that resulted in the incorrect identification of the required behavior of the Prandtl number. Also the limiting boundary conditions were not correctly described in ref. 6 (the origin of that mistake might lie in ref. 11 where the self-adjoint boundary conditions were analyzed). We find in the fine-tuned limit $l_\nu, l_\kappa \to 0$ the different, so called *sticky* or *slowly reflecting*, boundary conditions [2,20] parametrized by the amount of "glue" $\lambda \in [0, \infty]$.

The sticky behavior can be pictured intuitively in the following way. For simplicity let us consider just the case of Brownian motion on the half-line. Concerning ordinary reflected Brownian motion, it is a well known fact that, if it hits the end-point 0 of the half-line at some instant t, then for any instant $t' > t$ it will hit again 0 between t and t' an infinite number of times almost surely, but still the cumulated time spent at zero is zero almost surely. Slowly reflected Brownian motion returns also almost surely an infinite number of times after the first hit of the boundary, but the cumulated time spent at zero (technically speaking the Lebesgue measure of those instants in $[t, t']$ when the particle is at zero) will be proportional to the time spent in an infinitesimal neighborhood $(0, \mathrm{d}x)$ of 0 (not containing 0!), the proportionality constant being $\lambda/\mathrm{d}x$. Remarkably, however, for any $\lambda < \infty$, the trajectory will almost surely never remain at 0 for an uninterrupted strictly positive time-interval. The two extreme values of λ are $\lambda=0$ and $\lambda=\infty$. In the first case zero time is spent at zero and, in fact, one recovers the ordinary (instantaneous) reflection. The second case with infinite time spent at zero corresponds to absorption (more precisely, adsorption) at zero. Transposed to the case of Lagrangian dispersion, this means that particles upon hitting each other either have no interaction (case $\lambda=0$), or they will tend to stay together for some positive amount of cumulated time (case $0<\lambda<0$), or they stick together permanently (case $\lambda=\infty$).

We shall give strong general arguments in favor of the convergence of the particle dispersion to the sticky process and shall prove the convergence for specific quantities. To analyze the effect of stickiness on the passive transport, we shall construct and analyze the behavior of the transition probabilities of the sticky dispersion process that govern, in the appropriate $l_\nu, l_\kappa \to 0$ limit, the evolution of the 2-point correlation function of advected tracer in the homogeneous and isotropic situation. It appears that the presence of stickiness induces anomalous scaling of the stationary 2-point structure function of the forced tracer and that it influences the rate of tracer energy condensation in the constant mode. For the two extreme cases of instantaneous reflection and absorption, not only the particle dispersion process but the entire family of the consistent N-particle Markov processes has been constructed in refs. 16 and 17. In the sticky case, the existence and the uniqueness of such a construction is still an open problem and is not discussed further here. Let us only note that ref. 18 may be interpreted as providing such a construction for the limiting case of the one-dimensional flow with $\xi=0$.

Let us recapitulate the situation that we are dealing with. We are studying the intermediate compressibility regime of the Kraichnan model of passive advection with velocity field smoothed at scales smaller than l_ν and with molecular diffusivity κ dominant on scales smaller than l_κ. We shall be looking at the 2-point function of the advected tracer in the isotropic sector or, alternatively, at the PDF of the distance between two noisy Lagrangian trajectories. We want to know how those objects evolve in the limit $l_\nu, l_\kappa \to 0$. The remainder of the paper is organized as follows. In Section 3 we shall discuss the probabilistic aspects of the problem studying the one-dimensional diffusion process describing the inter-particle distance. The analytic approach based on the spectral analysis of the generator of the process will be developed in Section 4. The analysis of the limits $l_\nu, l_\kappa \to 0$ in those sections is based on approximate calculations. In Section 5, we study the sticky limiting behavior of the trajectories obtained this way in more detail and we analyze its implications for the tracer transport. To confirm the approximate analysis of the first sections, we rigorously show in Section 6 that, in the intermediate compressibility regime, the stationary 2-point structure function of the forced tracer indeed converges when $l_\nu, l_\kappa \to 0$ in the fine-tuned way to the one corresponding to the sticky behavior of Lagrangian trajectories. Finally, in Conclusions, we summarize the obtained results and mention some open problems that they raise. Appendices contain more technical material relevant for the main text.

3. STOCHASTIC PROCESS VIEWPOINT

3.1. Natural Scale and Speed Measure

In the intermediate compressibility regime, zero is a regular boundary for the unregularized operator M of Eq. (17) according to the Feller criteria. That is, if M is viewed as the generator of a stochastic process $r(t)$ on $\mathbb{R}_+$ then the realizations of the process may hit zero and come back from it, with positive probability. In order to completely describe the process, one needs to specify a boundary condition for M at zero. On the other hand, for l_ν, l_κ finite, the regularized generator $M_{\nu,\kappa}$ of Eq. (13) should be taken with Neumann boundary condition at zero. As we have discussed above, this is because the problem on the half-line arose as the rotationally invariant sector of a non-degenerate problem defined on the d-dimensional space.

To see what boundary condition is obtained when $l_\nu, l_\kappa \to 0$, it is useful to study the natural scale and the speed measure (see Appendix A) of the process $r(t)$ with positive l_ν, l_κ and to determine their behavior in the aforementioned limit. The natural scale $S(r)$ is the (positively oriented) coordinate in which the generator of the diffusion $r(t)$ is without drift so that $S(r(t))$ is a martingale (there is of course an equivalence class of such coordinates, related by affine transformations). $S(r)$ is strictly increasing and continuous in r, so that the function $s([r_1, r_2]) \equiv S(r_2) - S(r_1)$ defined on intervals can be extended to a measure on $\mathbb{R}_+$ absolutely continuous w.r.t. the Lebesgue measure). The measure $\mathrm{d}s$ is sometimes called the natural scale measure. Often it is more practical to work with the density of a measure than the measure itself. For a measure $\mathrm{d}\mu$ and a coordinate r on $\mathbb{R}_+$ we denote by $\mu(r)$ the density of $\mathrm{d}\mu$ with respect to r. In the case of $\mathrm{d}s$ we have in particular $s(r) = \frac{\mathrm{d}S(r)}{\mathrm{d}r}$.

In terms of the functions $f_1(r)$, $f_2(r)$ entering the generator $M_{\nu,\kappa}$ one can choose for the density with respect to r of the natural scale measure

$$s(r) = \exp\left(- \int_{r_0}^{r} \frac{f_2(r')}{f_1(r')} \, \mathrm{d}r' \right) \tag{22}$$

where $r_0 > 0$ is an arbitrary point of the open half-line $\mathbb{R}_+^*$. It is easily checked that in the corresponding coordinate S, the process becomes driftless (no first order derivative):

$$M_{\nu,\kappa} = \tilde{D}_0\, f_1(r)\, s(r)^2 \partial_S^2 \;\equiv\; \tilde{D}_0\, m(S)^{-1} \partial_S^2 .$$

We shall define the speed measure as the measure $\mathrm{d}m$ with density $m(S)$ with respect to the coordinate S. For convenience, we have multiplied the speed measure by $2\tilde{D}_0$ with respect to the conventions of Appendix A in order to avoid such factors in later expressions. The diffusion process $S(t) = S(r(t))$ is then a (random, i.e. realization dependent) re-parametrization of the Brownian motion $W(\tau)$, i.e. $S(t) = W(\tau(t))$, with

$$2\tilde{D}_0 \frac{\mathrm{d}t}{\mathrm{d}\tau} = m(S)|_{S=W(\tau)} .$$

For later use we also calculate the density of the speed measure with respect to the coordinate r:

$$m(r) = m(S)\frac{\mathrm{d}S}{\mathrm{d}r} = [f_1(r)s(r)]^{-1} = \frac{1}{f_1(r)} \exp\left(\int_{r_0}^{r} \frac{f_2(r')}{f_1(r')}\,\mathrm{d}r'\right) \tag{23}$$

with the same r_0 as for the natural scale above.

We expect that the process obtained in the $l_\kappa, l_\nu \to 0$ limit corresponds to the natural scale and the speed measure that are appropriate limits of the same objects for l_κ, l_ν finite. In the next subsections, we shall analyze those limits in an approximate way.

3.2. Approximate Calculations

We should know the dependence of functions f_1 and f_2 on the regularization scales l_ν, l_κ. If $l_\nu < l_\kappa$ then the smoothing of the velocity field is only significant at scales where it is already not the advection but the diffusion term that dominates, that is to say, the smoothing of the velocity plays no role and we may take $l_\nu = 0$ immediately. This case goes without difficulties and has been studied in ref. 10, see also ref. 5. The case which is interesting for us is the other one, when $l_\kappa < l_\nu$. In this case we feel the diffusivity between 0 and l_κ, then we feel the smoothing of the velocity field between l_κ and l_ν and finally above l_ν we are in the genuine Kraichnan regime with some scaling exponent ξ of the second order velocity structure function.

In the present section, the calculations will be made by replacing functions f_1, f_2 by functions glued piecewise from pure powers, representing the different scaling behaviors in different subintervals. The gluing is done so that the function f_1 stays continuous. That the model functions obtained this way are indeed correct approximations of functions f_1, f_2 for the regularized Kraichnan model is shown in Appendix F. Thus for $r > l_\nu$ we shall take $f_1(r)$ and $f_2(r)$ as in the (scale-invariant) Kraichnan model, that is $f_1(r) = r^\xi$ and $f_2(r) = a_\xi r^{-1} f_1(r)$ with a_ξ defined in (18). For $l_\kappa < r < l_\nu$ the functions f_1, f_2 will behave as in the smooth Kraichnan model (i.e. for $\xi = 2$), and we have to match them at l_ν with the values already given above (approximately; what matters, as we shall see, is only the order of magnitude and the ratio of the functions). Thus we shall take $f_1(r) = r^2 l_\nu^{\xi-2}$ and $f_2(r) = a_2 r^{-1} f_1(r)$. In the same way for $r < l_\kappa$ the functions f_1 and f_2 will be like in the pure diffusive case which, incidentally, is the same as the Kraichnan model with $\xi = 0$. With the same kind of matching as above, but now at l_κ, we get $f_1(r) = l_\kappa^2 l_\nu^{\xi-2}$ and $f_2(r) = (d-1) r^{-1} f_1(r)$ there. Table II sums up our choices for the approximate versions of f_1 and f_2.

With the pure power choices for f_1 and f_2, it is straightforward to calculate the natural scale and the speed measure on the intervals $[0, l_\kappa], [l_\kappa, l_\nu]$ and $[l_\nu, \infty)$. Let us first evaluate the ubiquitous subexpression $s(r) = \exp(-\int_{r_0}^{r} f_2(r')/f_1(r')\,\mathrm{d}r')$. We see that in each of the above intervals, $f_2(r')/f_1(r') = a/r'$ with some a constant on the interval, so the integral evaluates to logarithms. Again, results are summed up in Table III.

Table II. Approximate Versions of Functions f_1 and f_2

Scale	$f_1(r)$	$rf_2(r)/f_1(r)$
$r \in [0, l_\kappa]$	$l_\kappa^2 l_\nu^{\xi-2}$	$d-1$
$r \in [l_\kappa, l_\nu]$	$r^2 l_\nu^{\xi-2}$	a_2
$r \in [l_\nu, \infty]$	r^ξ	a_ξ

Table III. Densities of the Natural Scale and Speed Measures

Scale	$s(r)$	$m(r)$
$r \in [0, l_\kappa]$	$(r_0/l_\nu)^{a_\xi} (l_\nu/l_\kappa)^{a_2} (l_\kappa/r)^{d-1}$	$r_0^{-\xi} (l_\nu/r_0)^{a_\xi-\xi} (l_\kappa/l_\nu)^{a_2-2} (r/l_\kappa)^{d-1}$
$r \in [l_\kappa, l_\nu]$	$(r_0/l_\nu)^{a_\xi} (l_\nu/r)^{a_2}$	$r_0^{-\xi} (l_\nu/r_0)^{a_\xi-\xi} (r/l_\nu)^{a_2-2}$
$r \in [l_\nu, \infty]$	$(r_0/r)^{a_\xi}$	$r_0^{-\xi} (r/r_0)^{a_\xi-\xi}$

3.3. Limit of the Natural Scale

It is easy to see that if we maintain $r_0 > 0$ fixed, then the density $s(r)$ of the natural scale measure tends pointwise to

$$s_0(r) \equiv r_0^{a_\xi} r^{-a_\xi} . \tag{24}$$

Recall that $s(r) = \frac{\mathrm{d}S(r)}{\mathrm{d}r}$. Thus $S(r)$ has to be the integral of $s(r)$, but we have the freedom to choose the constant of integration. We adopt the choice

$$S(r) \equiv \int_0^{r_0} s_0(r')\,\mathrm{d}r' + \int_{r_0}^{\mathrm{r}} \mathrm{s}(r')\,\mathrm{d}r' .$$

This definition has the advantage that, for $r > l_\nu$, the value of $S(r)$ is independent of l_κ, l_ν, because $s(r) = s_0(r)$ there. In fact we have

$$S(r) = S_0(r) \equiv \int_0^r s_0(r')\,\mathrm{d}r' = \frac{1}{1-a_\xi} r_0^{a_\xi} r^{1-a_\xi} \qquad \text{if} \quad r \geqslant l_\nu . \tag{25}$$

In particular, in the limit $l_\kappa, l_\nu \to 0$, the natural scale function becomes equal to $S_0(r)$ for $r > 0$. Note that $S_0(0) = 0$, observe, however, that if $d \geqslant 2$ then $S(0) = -\infty$ as long as l_κ is positive since in that case the integral $\int_{l_\kappa}^r s(r')\,\mathrm{d}r'$ diverges as r goes to zero. Relation (25) may be inverted to give

$$r(S) = [(1-a_\xi) r_0^{-a_\xi} S]^{\frac{1}{1-a_\xi}} \geqslant l_\nu \qquad \text{if} \quad S \geqslant \frac{1}{1-a_\xi} r_0^{a_\xi} l_\nu^{1-a_\xi} . \tag{26}$$

3.4. Limit of the Speed Measure

Let us now calculate the speed measure of each of the intervals $[0, l_\kappa]$, $[l_\kappa, l_\nu]$ and for the sake of completeness $[l_\nu, R]$ for arbitrary $R > l_\nu$.

For $[0, l_\kappa]$ we get

$$\begin{aligned} m([0, l_\kappa]) &= \int_0^{l_\kappa} m(r)\,\mathrm{d}r = \int_0^{l_\kappa} \frac{1}{l_\kappa^2 l_\nu^{\xi-2}} \left(\frac{l_\nu}{r_0}\right)^{\mathrm{a}_\xi} \left(\frac{l_\kappa}{l_\nu}\right)^{\mathrm{a}_2} \left(\frac{r}{l_\kappa}\right)^{\mathrm{d}-1} \mathrm{d}r \\ &= \frac{r_0^{-a_\xi}}{d} l_\nu^{(a_\xi+1-\xi)+(1-a_2)} l_\kappa^{a_2-1} . \end{aligned}$$

In the intermediate compressibility regime, $a_\xi + 1 - \xi > 0$ and $1 - a_2 > 0$, hence we have a positive power of l_ν and a negative power of l_κ. Depending on the way in which l_ν, l_κ go to zero, $m([0, l_\kappa])$ can tend to zero, infinity or some fixed constant. For the finite limit one should have $l_\nu^{(a_\xi+1-\xi)+(1-a_2)} \propto l_\kappa^{1-a_2}$, i.e.

$$l_\kappa \propto l_\nu^{1+\frac{a_\xi+1-\xi}{1-a_2}} \tag{27}$$

The exponent of l_ν being greater than 1, this relation is compatible with the condition $l_\kappa \ll l_\nu$ in the limit $l_\nu, l_\kappa \to 0$. In a moment we shall explain how the limit $l_\nu, l_\kappa \to 0$ taken with condition (27) corresponds to the sticky boundary condition. For now let us simply give some equivalent formulations of (27). From (14), it can be seen that $2\kappa = \tilde{D}_0 f_1(0)$. Within our current modeling of f_1, we have $f_1(0) = l_\nu^{\xi-2} l_\kappa^2$ so that

$$\kappa = \frac{1}{2}\tilde{D}_0\, l_\nu^{\xi-2}\ l_\kappa^2. \tag{28}$$

We shall use this equation to fix the relation between κ and l_κ also in the exact approach. Recalling definition (7), we see that the Prandtl number $Pr \equiv \frac{\nu}{\kappa}$ is proportional to $(l_\nu/l_\kappa)^2$ for $l_\kappa \ll l_\nu$ so that imposing relation (27) is equivalent to

$$\kappa \propto l_\nu^{\xi+2\frac{a_\xi+1-\xi}{1-a_2}} \qquad \text{or} \qquad Pr \propto l_\nu^{-2\frac{a_\xi+1-\xi}{1-a_2}}.$$

In particular κ goes to zero while Pr goes to infinity when l_ν goes to zero.

The calculation of the speed measure of the interval $[l_\kappa, l_\nu]$ is performed similarly:

$$\begin{aligned} m([l_\kappa, l_\nu]) &= \int_{l_\kappa}^{l_\nu} m(r)\,\mathrm{d}r = \int_{l_\kappa}^{l_\nu} \frac{1}{r^2 l_\nu^{\xi-2}}\left(\frac{l_\nu}{r_0}\right)^{a_\xi}\left(\frac{r}{l_\nu}\right)^{a_2}\mathrm{d}r \\ &= \frac{r_0^{-a_\xi}}{a_2-1} l_\nu^{a_\xi-a_2+2-\xi}\left(l_\nu^{a_2-1} - l_\kappa^{a_2-1}\right). \end{aligned}$$

Now, since in the intermediate compressibility regime $a_2 - 1 < 0$, in the limit $l_\nu, l_\kappa \to 0$ it will be $l_\kappa^{a_2-1}$ that dominates $l_\nu^{a_2-1}$ so that

$$m([l_\kappa, l_\nu]) \approx \frac{r_0^{-a_\xi}}{1-a_2} l_\nu^{(a_\xi+1-\xi)+(1-a_2)} l_\kappa^{a_2-1}.$$

We are in exactly the same situation as above and the same conclusions hold.

Finally the speed measure of the interval $[l_\nu, R]$ for some arbitrary $R > l_\nu$ is

$$m([l_\nu, R]) \;=\; \int_{l_\nu}^{R} m(r)\,\mathrm{dr} \;=\; \int_{l_\nu}^{R} r^{-\xi}\left(\frac{r}{r_0}\right)^{a_\xi} \mathrm{d}r$$
$$= \frac{r_0^{-a_\xi}}{a_\xi+1-\xi}\left(R^{a_\xi+1-\xi} - l_\nu^{a_\xi+1-\xi}\right) \approx \frac{r_0^{-a_\xi}}{a_\xi+1-\xi}R^{a_\xi+1-\xi}$$

and it tends to a finite limit when $l_\nu, l_\kappa \to 0$.

At this point we may describe explicitly the speed measure $\mathrm{d}m_0$ obtained as the weak limit of speed measures for positive l_ν, l_κ when $l_\nu, l_\kappa \to 0$ in such a way that

$$l_\nu^{(a_\xi+1-\xi)+(1-a_2)} l_\kappa^{a_2-1} \;\to\; \lambda \tag{29}$$

for some $\lambda \in [0, +\infty]$ (note that λ has dimension $(\mathrm{length})^{a_\xi+1-\xi}$). On the open half-line of $r > 0$ the density of $\mathrm{d}m_0$ is

$$m_0(r) = r_0^{-a_\xi} r^{a_\xi-\xi}. \tag{30}$$

Besides, $\mathrm{d}m_0$ has a mass at zero, given by

$$m_0(\{0\}) = \lim_{l_\nu, l_\kappa \to 0}(m([0, l_\kappa]) + m([l_\kappa, l_\nu])) = \frac{d+1-a_2}{(1-a_2)d}\, r_0^{-a_\xi}\,\lambda\,. \tag{31}$$

It is convenient to characterize the "stickiness" of the boundary at $r=0$ by the quantity $\tilde{\lambda} \equiv m_0(\{0\})/[r^{\xi-a_\xi} m_0(r)]_{r=0}$ which we shall call the *glue parameter*. We obtain

$$\tilde{\lambda} = \frac{d+1-a_2}{(1-a_2)d}\lambda \tag{32}$$

This relation will be compared to the value produced by another approach in the next section, and finally to the exact result (62) calculated using the precise forms of f_1, f_2 instead of the approximate versions of Table II.

3.5. Convergence on the Natural Scale

Here we shall show that the operations of changing coordinates from the original scale r to natural scale S and of taking the weak limit of the speed measure $\mathrm{d}m$ when $l_\nu, l_\kappa \to 0$ commute. Observe that for $d = 1$ the natural scale coincides with the original one so we shall be preoccupied here only by the case $d \geqslant 2$.

In the preceding subsection we computed the limit of $m(r)$, working on the original scale r. Here we shall pass first to the natural scale S and then take the limit of $m(S)$ and finally check that the result agrees with that of the preceding subsection. This is in order to exclude that we are in the kind of pathological case presented in Appendix B. A priori such a situation could arise here because the notions of weak convergence on the original scale and on the natural scale do not coincide.

The density of the speed measure w.r.t. the natural scale S is $m(S) = m(r)s^{-1}(r)$. From (26) it is clear that for any $S>0$ we can take l_ν sufficiently small to have $r(S)>l_\nu$ and thus $m(r)=m_0(r)$ and $s(r)=s_0(r)$ so that $m(S)=m_0(r)s_0^{-1}(r)=m_0(S_0)$. Hence the trivial convergence of the density of the speed measure on the open half-line of positive S to $m_0(S_0)$ in agreement with the result of the preceding subsection.

We still have to see what happens for $S<0$ and to show that the limit $m_0(S_0)$ of the density of the speed measure has a Dirac delta term at $S_0 = 0$. Let us evaluate for $S<0$ the asymptotic behavior of $r(S)$ as $l_\kappa, l_\nu \to 0$. Observe that $S(l_\nu)=\int_0^{l_\nu} s_0(r')\,\mathrm{d}r' > 0$. So if $S<0$ then $r(S)<l_\nu$. Now

$$\begin{aligned} S(l_\kappa) &= \int_0^{l_\nu} s_0(r')\,\mathrm{dr}' - \int_{l_\kappa}^{l_\nu} s(r')\,\mathrm{d}r' \\ &= \frac{1}{1-a_\xi} r_0^{a_\xi} l_\nu^{1-a_\xi} - \frac{1}{1-a_2} r_0^{a_\xi} l_\nu^{1-a_\xi} \left[1-\left(\frac{l_\kappa}{l_\nu}\right)^{1-a_2}\right] \longrightarrow 0 \end{aligned}$$

when $l_\nu \to 0$ with $l_\kappa < l_\nu$ since both $1-a_\xi$ and $1-a_2$ are positive. This shows that for $S<0$ fixed, we must have $r(S)<l_\kappa$ asymptotically. Let us explicitly treat the case $d>2$. The case $d=2$ is only a bit different because of the logarithmic divergences. For $r<l_\kappa$,

$$\begin{aligned} S(r) &= S(l_\kappa) - \int_r^{l_\kappa} s(r')\,\mathrm{d}r' \\ &= S(l_\kappa) - \frac{1}{2-d} r_0^{a_\xi} l_\nu^{a_2-a_\xi} l_\kappa^{d-1-a_2} (l_\kappa^{2-d} - r^{2-d}). \end{aligned}$$

Suppose that $r=o(l_\kappa)$ (consistency will be checked) so that $l_\kappa^{2-d}=o(r^{2-d})$. Recalling that $S(l_\kappa)\to 0$, we infer that, for fixed $S<0$,

$$\begin{aligned} r(S) &\sim \left[(2-d)\, r_0^{-a_\xi} l_\nu^{a_\xi-a_2} l_\kappa^{1-d+a_2} S\right]^{\frac{1}{2-d}} \\ &= \left[(d-2)\, r_0^{-a_\xi}(-S)\right]^{-\frac{1}{d-2}} \left[l_\nu^{1-a_\xi}(l_\kappa/l_\nu)^{1-a_2}\right]^{\frac{1}{d-2}} l_\kappa. \end{aligned}$$

We immediately check that $r(S) = o(l_\kappa)$ so that our estimates are consistent, as promised. It can also be seen that this asymptotics for $r(S)$ is uniformly valid on $(-\infty, S]$ for any $S < 0$. Now we may write

$$m(S) = [m(r)\, s^{-1}(r)]_{r=r(S)}$$
$$\sim \Big[(d-2)^{-(d-1)}\, r_0^{a_\xi}\, (l_\kappa / l_\nu)^{1-a_2}\, l_\nu^{1-a_\xi}\Big]^{\frac{2}{d-2}}\, l_\nu^{2-\xi}\, (-S)^{-[2+\frac{2}{d-2}]}.$$

We infer that $m(S)$ decays as a power of $-S$ when $S \to -\infty$, fast enough to be integrable. We also see that the coefficient in front of the power of $-S$ goes to zero when $l_\nu \to 0$ with $l_\kappa < l_\nu$.

We still have to show that $m_0(S)$ has a Dirac delta contribution at zero. This can now be seen because

$$\int_{-\infty}^{S(l_\nu)} m(S)\, \mathrm{dS} = \int_0^{l_\nu} m(r)\, \mathrm{d}r$$

and we have shown that the right hand side has the limit given by (31). On the other hand we have just seen that for any $S < 0$

$$\int_{-\infty}^{S} m(S')\, \mathrm{d}S' \;\longrightarrow\; 0$$

and also that $S(l_\nu) > 0$ and $S(l_\nu) \to 0$. This permits to conclude that $m_0(S_0)$ has a Dirac delta contribution at $S_0 = 0$ with the coefficient $m_0(\{0\})$. An analogous reasoning for the case $d = 2$ gives the same result regarding the (weak) convergence of the measure $\mathrm{d}m$ on the natural scale. The limit measure coincides with the limiting speed measure in the r coordinate re-expressed in the limiting natural scale coordinate S_0.

3.6. Conjecture

It is legitimate to expect that the process $r(t)$ converges in an appropriate sense to the one corresponding to the limit natural scale function $S_0 \geqslant 0$ given in (25), and the speed measure with density

$$m_0(S_0) = r_0^{-a_\xi / b_\xi}\, [(1-a_\xi) S_0]^{-2+\frac{1}{b_\xi}} \;+\; m_0(\{0\})\, \delta(S_0)\,.$$

Such a process is, in the natural scale coordinate S_0, a reparametrization of the Brownian motion $|W(\tau)|$ instantaneously reflecting at zero: $S_0(t) =$

$|W(\tau(t))|$, see Appendix A. Away from zero, $2\tilde{D}_0 \frac{dt}{d\tau} = m_0(S_0)|_{S_0=|W(\tau(t))|}$. At zero, the reparametrized process spends time proportional to $m_0(\{0\})$ times the local time at zero of $|W(\tau)|$ (although never an uninterrupted open interval of time). This is the "sticky" or "slowly reflecting" boundary behavior(3,22) with the glue parameter $\tilde{\lambda} \propto m_0(\{0\})$, see (32). The extreme cases $\tilde{\lambda} = 0$ and $\tilde{\lambda} = \infty$ correspond, respectively, to instantaneous reflection and total absorption of the process at zero.

This was our first argument for the sticky boundary behavior.

4. LINEAR OPERATOR VIEWPOINT

Our next argument is based on the study of the (generalized) eigenfunctions of the operator $M_{\nu,\kappa}$ of Eq. (13) in the limit $l_\nu, l_\kappa \to 0$. If we take f_1, f_2 composed as above from pure powers, then we are able to calculate exactly the eigenfunctions and trace their limiting behavior. This argument is somewhat more shaky because it needs some explicit, not totally realistic (though representative), form of the functions f_1 and f_2 (before, such a representation was really required to hold only approximately).

4.1. Derivation of the Boundary Condition at Zero

Let us consider the differential operator $\tilde{D}_0^{-1} M_{\nu,\kappa} = f_1(r)\partial_r^2 + f_2(r)\partial_r$ with the approximate form of functions f_1 and f_2 given in Table II. Thus for $r > l_\nu$ we take $f_1(r) = r^\xi$ and $f_2(r) = a_\xi r^{-1} f_1(r)$ so on the interval $[l_\nu, \infty]$ a pair of linearly independent eigenfunctions corresponding to the eigenvalue $-E$ of $\tilde{D}_0^{-1} M_{\nu,\kappa}$ is

$$\Phi_E^{\pm}(r) = r^{\frac{1-a_\xi}{2}} J_{\pm b_\xi}\left(\sqrt{E}\, r^{\frac{2-\xi}{2}}\right)$$

where b_ξ is given by (21) and J_b is the Bessel function of the first kind of parameter b. Note that these eigenfunctions are independent of l_ν, l_κ. When we impose at zero a boundary condition,[5] in general the eigenfunctions of the operator with boundary condition will form a one dimensional linear subspace of the two dimensional linear space spanned by $\Phi_E^{\pm}$. That is, if the linear combination $\tilde{c}_E^+ \Phi_E^+ + \tilde{c}_E^- \Phi_E^-$ is an eigenfunction verifying the boundary condition at zero, then $\tilde{c}_E^+ / \tilde{c}_E^-$ is fixed (depending on E and the boundary condition, of course). In part we shall proceed in the

[5] Exactly one, given as $L[\Phi_E] = 0$ where L is an operator that can be expressed as the limit at zero of some finite linear combination of derivatives of order 0 or higher, with not necessarily constant coefficients.

opposite way. First we calculate $\tilde{c}_E^+/\tilde{c}_E^-$ for l_ν, l_κ finite. Next we deduce the effective boundary condition at zero that would give the same quotient if l_ν, l_κ were zero. Finally we look at the limit of $\tilde{c}_E^+/\tilde{c}_E^-$ when $l_\kappa, l_\nu \to 0$, and deduce the "limit" of the effective boundary condition.

First of all we give explicitly a generating pair of eigenfunctions of $\tilde{D}_0^{-1} M_{\nu,\kappa}$ associated with the eigenvalue $-E$, on each of the intervals $[0, l_\kappa], [l_\kappa, l_\nu], [l_\nu, \infty]$. We shall denote these six functions $g_1^\pm, g_2^\pm, g_3^\pm$ respectively. Because at zero we have the Neumann boundary condition, on $[0, l_\kappa]$ we give only the corresponding eigenfunction:

$$g_1^-(r) = C_1 r^{\frac{2-d}{2}} J_{-\frac{2-d}{2}}\left(\sqrt{E/\kappa}\, r\right) \quad \text{with} \quad C_1 = 2^{-\frac{2-d}{2}} \Gamma\left(1 - \frac{2-d}{2}\right) (E/\kappa)^{\frac{2-d}{4}}$$

(the normalization was chosen for later convenience). On $[l_\kappa, l_\nu]$ we may set

$$g_2^\pm(r) = r^{\gamma_\pm(E, l_\nu)} \quad \text{where} \quad \gamma_\pm(E, l_\nu) = \frac{1 - a_2 \pm \sqrt{(1-a_2)^2 - 4 l_\nu^{2-\xi} E}}{2}$$

(recall that $1 - a_2 > 0$ in the intermediate compressibility regime). Finally on $[l_\nu, \infty]$ we take

$$g_3^\pm(r) = 2^{\pm b_\xi} \Gamma(1 \pm b_\xi) E^{\mp \frac{b_\xi}{2}} \Phi_E^\pm(r)\,.$$

In order to construct the eigenfunction of $M_{\nu,\kappa}$ on the whole half-line $[0, \infty]$, we have to find the correct linear combinations of the generating pairs on each sub-interval by matching the functions and their derivatives at each border point. Formally, if we have to match $c_i^+ g_i^+ + c_i^- g_i^-$ with $c_j^+ g_j^+ + c_j^- g_j^-$ at r, then, in matrix notation, we should have

$$\begin{pmatrix} g_i^+(r) & g_i^-(r) \\ g_i^{+\prime}(r) & g_i^{-\prime}(r) \end{pmatrix} \begin{pmatrix} c_i^+ \\ c_i^- \end{pmatrix} = \begin{pmatrix} g_j^+(r) & g_j^-(r) \\ g_j^{+\prime}(r) & g_j^{-\prime}(r) \end{pmatrix} \begin{pmatrix} c_j^+ \\ c_j^- \end{pmatrix}$$

where $g'(r) \equiv \partial_r g(r)$. That is to say,

$$\begin{pmatrix} c_j^+ \\ c_j^- \end{pmatrix} = \begin{pmatrix} g_j^+(r) & g_j^-(r) \\ g_j^{+\prime}(r) & g_j^{-\prime}(r) \end{pmatrix}^{-1} \begin{pmatrix} g_i^+(r) & g_i^-(r) \\ g_i^{+\prime}(r) & g_i^{-\prime}(r) \end{pmatrix} \begin{pmatrix} c_i^+ \\ c_i^- \end{pmatrix}.$$

Now, because on $[0, l_\kappa]$ we take g_1^- as the eigenfunction, it follows that, by making the above described matchings, we get on $[l_\nu, \infty]$ the linear combination $c_E^+ g_3^+ + c_E^- g_3^-$ with

$$\begin{pmatrix} c_E^+ \\ c_E^- \end{pmatrix} = \begin{pmatrix} g_3^+(l_\nu) & g_3^-(l_\nu) \\ g_3^{+\prime}(l_\nu) & g_3^{-\prime}(l_\nu) \end{pmatrix}^{-1} \begin{pmatrix} g_2^+(l_\nu) & g_2^-(l_\nu) \\ g_2^{+\prime}(l_\nu) & g_2^{-\prime}(l_\nu) \end{pmatrix} \begin{pmatrix} g_2^+(l_\kappa) & g_2^-(l_\kappa) \\ g_2^{+\prime}(l_\kappa) & g_2^{-\prime}(l_\kappa) \end{pmatrix}^{-1} \begin{pmatrix} g_1^-(l_\kappa) \\ g_1^{-\prime}(l_\kappa) \end{pmatrix}. \quad (33)$$

Note that

$$\frac{\tilde{c}_E^+}{\tilde{c}_E^-} = \frac{\Gamma(1+b_\xi)}{\Gamma(1-b_\xi)} \left(\frac{4}{E}\right)^{b_\xi} \frac{c_E^+}{c_E^-}.$$

For a general matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its inverse is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We use this formula to rewrite (33), but since we are only interested in the ratio c_E^+/c_E^-, we get rid of the $\frac{1}{ad-bc}$ factor. Thus

$$\begin{pmatrix} c_E^+ \\ c_E^- \end{pmatrix} \propto \begin{pmatrix} g_3^{-\prime}(l_\nu) & -g_3^-(l_\nu) \\ -g_3^{+\prime}(l_\nu) & g_3^+(l_\nu) \end{pmatrix} \begin{pmatrix} g_2^+(l_\nu) & g_2^-(l_\nu) \\ g_2^{+\prime}(l_\nu) & g_2^{-\prime}(l_\nu) \end{pmatrix} \begin{pmatrix} g_2^{-\prime}(l_\kappa) & -g_2^-(l_\kappa) \\ -g_2^{+\prime}(l_\kappa) & g_2^+(l_\kappa) \end{pmatrix} \begin{pmatrix} g_1^-(l_\kappa) \\ g_1^{-\prime}(l_\kappa) \end{pmatrix}. \quad (34)$$

This gives an explicit formula for c_E^+/c_E^- for l_κ, l_ν finite. In Appendix C we show that if we take l_ν, l_κ to zero in a way that the limit (29) exists (notation $l_\nu, l_\kappa \xrightarrow{\lambda} 0$) then

$$\lim_{l_\nu, l_\kappa \xrightarrow{\lambda} 0} \frac{c_E^+}{c_E^-} = -\frac{d+1-a_2}{(1-a_2)(1-a_\xi)d}\, \lambda\, E. \quad (35)$$

It is not difficult to identify the boundary condition implied by this relation. Recall that the eigenfunctions of $\tilde{D}_0^{-1} M_{\nu,\kappa}$ associated to the eigenvalue $-E$ are of the form $\Phi_E(r) = c_E^+ g_3^+(r) + c_E^- g_3^-(r)$ for $r > l_\nu$, (that is for $r > 0$ if $l_\nu, l_\kappa \to 0$). Furthermore we see that

$$g_3^-(0) = 1, \qquad r^{a_\xi} g_3^{-\prime}(r)|_{r=0} = 0,$$
$$g_3^+(0) = 0, \qquad r^{a_\xi} g_3^{+\prime}(r)|_{r=0} = 1 - a_\xi.$$

From this we conclude that Φ_E verifies

$$\Phi_E(0) = c_E^- \qquad \text{and} \qquad r^{a_\xi} \Phi_E'(r)|_{r=0} = (1-a_\xi) c_E^+. \quad (36)$$

For convenience let us denote $\tilde{D}_0 r^{a_\xi} \partial_r \equiv F$ and, as before, $M_{0,0} \equiv M$. Now (36) with (35) imply that

$$\tilde{D}_0^{-1}(F\Phi_E)(0) = -\tilde{\lambda}\, E\, \Phi_E(0)$$

where $\tilde{\lambda}$ is given by (32). Since Φ_E is an eigenvector of $\tilde{D}_0^{-1}M$ with eigenvalue $-E$, we obtain

$$(F\Phi_E)(0) = \tilde{\lambda}\,(M\Phi_E)(0)\,. \tag{37}$$

This last form may be read as an independent of E boundary condition that all Φ_E verify. It becomes then *the* boundary condition for M. Note that the coefficient $\tilde{\lambda}$ is in $[0,+\infty]$. We introduce thus the operator $M_{\tilde{\lambda}}$ which is the version of M with the domain $\mathcal{D}(M_{\tilde{\lambda}})$ corresponding to the boundary condition (37). In the next subsection we shall show that $M_{\tilde{\lambda}}$ is the Kolmogorov backwards generator of a sticky process whose glue parameter is equal to $\tilde{\lambda}$.

4.2. Interpretation of the Boundary Condition

To see that boundary condition (37) corresponds to a slowly reflecting boundary, we may calculate the mass at zero of the speed measure of the diffusion process generated by $M_{\tilde{\lambda}}$. The speed measure may be defined, up to a multiplicative constant, as the measure $\mathrm{d}m$ with respect to which the generator $M_{\tilde{\lambda}}$ is self-adjoint.[20] It is straightforward to see that for $r>0$ we may take for the density of the speed measure with respect to r (recall notation of Section 3)

$$m(r) = r^{a_\xi - \xi}\,. \tag{38}$$

We have in particular the relation $m(r)M = \tilde{D}_0\partial_r \circ r^{a_\xi}\,\partial_r$. Here, as in the following, if we write M instead of $M_{\tilde{\lambda}}$, we mean not the operator with a specified domain but only the corresponding differential expression. Let us denote the L^2 scalar product with respect to the measure $\mathrm{d}m$ by $(.\,,.)_m$. For $M_{\tilde{\lambda}}$ to be symmetric with respect to $\mathrm{d}m$, we need that $(g_1, Mg_2)_m = (Mg_1, g_2)_m$ for any $g_1, g_2 \in \mathcal{D}(M_{\tilde{\lambda}})$. Using (38) for the density of $\mathrm{d}m$ on the open half-line $(0,\infty)$, and putting in a separate term the effect of a possible mass of $\mathrm{d}m$ at zero, this may be written as

$$\begin{aligned} g_1(0)(Mg_2)(0)\,m(\{0\}) &+ \int_{0^+}^{\infty} g_1(r)(Mg_2)(r)\,m(r)\,\mathrm{d}r \\ = (Mg_1)(0)g_2(0)\,m(\{0\}) &+ \int_{0^+}^{\infty} (Mg_1)(r)g_2(r)\,m(r)\,\mathrm{d}r \end{aligned} \tag{39}$$

where the limit 0^+ in the integrals indicates that the integral is over the open half-line $(0,\infty)$. The integral on the left hand side may be rewritten using integration by parts as

$$\int_{0^+}^{\infty} g_1(r)(Mg_2)(r)\, m(r)\, \mathrm{d}r = g_1(0)(Fg_2)(0) - (Fg_1)(0)g_2(0) + \int_{0^+}^{\infty} (Mg_1)(r)g_2(r)\, m(r)\, \mathrm{d}r\,.$$

We then see that (39) holds if and only if

$$[(Mg_1)(0)\, m(\{0\}) - (Fg_1)(0)]\; g_2(0) \;=\; g_1(0)\; [(Mg_2)(0)\, m(\{0\}) - (Fg_2)(0)]$$

(strictly speaking the above expression is defined in the limit $r \to 0$). This condition may only be verified for all $g_1, g_2 \in \mathcal{D}(M_{\tilde{\lambda}})$ if, for all $g \in \mathcal{D}(M_{\tilde{\lambda}})$, $([m(\{0\})M - F]\, g)(0) = 0$. Recalling the boundary condition (37), this amounts to the equality

$$m(\{0\}) = \tilde{\lambda}\,. \tag{40}$$

We infer this way that the speed measure of $M_{\tilde{\lambda}}$ has a finite mass at zero, which means that $M_{\tilde{\lambda}}$ describes a process that is slowly reflecting at zero. From (40) and (38) we see that the glue parameter of the slow reflection is $\tilde{\lambda}$.

More generally, the speed measure found in this section is coherent with the one we got from the stochastic process treatment. Compare (30), (31) with (38), (40), respectively, to see that the two speed measures are proportional (the second equals $r_0^{a_\xi}$ times the first one).

5. STICKY PROCESS

Now that we know the effective time evolution operator on large scales, we may inquire how the corresponding transition probabilities of the Lagrangian dispersion and other related quantities look like, what is their long time behavior and the induced effect on tracer transport.

5.1. Transition Probabilities

The transition probabilities $P^t_{\tilde{\lambda}}(r; \mathrm{d}r')$ for the process with generator $M_{\tilde{\lambda}}$ are given by the self-adjoint exponential operators $e^{tM_{\tilde{\lambda}}}$ defined in $L^2(m(\mathrm{d}r))$. It will be, however, more convenient to use the more standard $L^2(\mathrm{d}r)$ conventions for the kernels and explicitly write the contribution due to the atomic term of $\mathrm{d}m$. In such notation,

$$P^t_{\tilde{\lambda}}(r; \mathrm{d}r') \;=\; e^{tM_{\tilde{\lambda}}}(r, r')\, \mathrm{d}r' + \tilde{\lambda}\left[e^{tM_{\tilde{\lambda}}}(r, \rho)\, \rho^{\xi - a_\xi}\right]_{\rho=0} \delta(r')\, \mathrm{d}r'\,. \tag{41}$$

The term concentrated at zero assures the conservation of probability $\int P^t_{\tilde\lambda}(r; \mathrm{d}r') = 1$ as is shown in Appendix E. Its presence implies that sticky trajectories spend together positive time. In particular, the mean of the time that two trajectories starting at moment zero at distance r spend together up to moment t is $\tilde\lambda \int_0^t [e^{sM_{\tilde\lambda}}(r, \rho)\, \rho^{\xi - a_\xi}]_{\rho=0}\, \mathrm{d}s$.

To compute the kernel $e^{tM_{\tilde\lambda}}(r, r')$, it will be more convenient to work with Schrödinger type operators. To this end, we shall first change variables by setting $u = \frac{2}{2-\xi}\tilde{D}_0^{-1/2} r^{\frac{2-\xi}{2}}$. Recall from (20) that this transforms M to the generator of a Bessel process of parameter $-b$ with b given by (21) (to ease the notation, we drop the subscript ξ). After passing to the new variable, we conjugate M by the multiplication by $u^{b-\frac{1}{2}}$ and finally multiply it by -1 to obtain the differential operator N of the form

$$N \equiv -\partial_u^2 + \frac{b^2 - \frac{1}{4}}{u^2}\,.$$

The boundary condition (37) imposed on $M_{\tilde\lambda}$ becomes for N the condition

$$u^{1-2b}\partial_u u^{b-\frac{1}{2}}\varphi(u)|_{u=0} = -\mu\, C_b^{-1}\, u^{b-\frac{1}{2}} N\varphi(u)|_{u=0} \tag{42}$$

where $C_b = 2^{2b-1}\frac{\Gamma(b)}{\Gamma(1-b)}$ is introduced for convenience, and

$$\mu \;=\; \frac{(2-\xi)^{2b-1}\,\Gamma(b)}{\Gamma(1-b)}\, \tilde{D}_0^{b-1}\,\tilde\lambda$$

is a positive constant proportional to $\tilde\lambda$. We shall denote by N_μ the operator N with this boundary condition. Note for later use the relationship

$$e^{tM_{\tilde\lambda}}(r, r')\,\mathrm{d}r' = u(r)^{b-\frac{1}{2}}\, e^{-tN_\mu}(u(r), u(r'))\, u(r')^{\frac{1}{2}-b}\,\mathrm{d}u(r') \tag{43}$$

The spectral properties of the operator N_μ are studied in detail in Appendix D. Here we rely on those results. The spectrum of N_μ is $\mathbb{R}_+$. The (generalized) eigenfunction $\varphi_{\mu,E}(u)$ of this operator, associated to the eigenvalue $E \geqslant 0$, may be taken as

$$\varphi_{\mu,E}(u) = u^{\frac{1}{2}}\left[J_{-b}\left(\sqrt{E}u\right) - \mu E^{1-b} J_b\left(\sqrt{E}u\right)\right]. \tag{44}$$

The (scalar) spectral measure corresponding to this choice (of normalization) of the eigenfunctions is

$$\mathrm{d}\nu_\mu(E) = \frac{\mathrm{d}E}{2(1-2\mu E^{1-b}\cos(b\pi)+\mu^2 E^{2(1-b)})}\,. \tag{45}$$

The kernel of the exponential of the operator N_μ may be written using spectral calculus as

$$\exp(-tN_\mu)(u,v) = \int_0^\infty e^{-tE}\,\varphi_{\mu,E}(u)\,\varphi_{\mu,E}(v)\,\mathrm{d}\nu_\mu(E)\,. \tag{46}$$

An important aspect of the flow is the behavior of two particles released at the same point at the same time. The evolution of the inter-particle distance of such a pair is described by the $r\to 0$ limit of $P^t_{\tilde\lambda}(r;\mathrm{d}r')$. Substituting (46) into (43) and that in turn into (41), we obtain:

$$\begin{aligned}\lim_{r\to 0}\; P^t_{\tilde\lambda}(r;\mathrm{dr}') = {}& \tfrac{2^b}{\Gamma(1-b)}\Big(\int_0^\infty e^{-tE}\,E^{-\frac{b}{2}}\varphi_{\mu,E}(u(r'))\,\mathrm{d}\nu_\mu(E)\Big)\,\mathrm{u}(\mathrm{r}')^{\frac{1}{2}-b}\,\mathrm{du}(r') \\ &+\frac{2(2-\xi)^{2b-1}\tilde D_0^{b-1}}{\Gamma(1-b)^2}\,\tilde\lambda\,\Big(\int_0^\infty e^{-Et}\,E^{-b}\,\mathrm{d}\nu_\mu(E)\Big)\,\delta(r')\,\mathrm{d}r'\,. \end{aligned} \tag{47}$$

The first term on the right hand side is the regular contribution absolutely continuous w.r.t. $\mathrm{d}r'$. It describes the probability that a pair of particles starting together at time zero are separated at time t by some finite distance r' with $\mathrm{d}r'$ precision. Presence of such a term indicates that the sticky Lagrangian flow should be stochastic in each velocity realization, just as the instantaneously reflecting flow corresponding to $\tilde\lambda=0$. The second term on the right hand side of (47) is concentrated at $r'=0$ and describes the probability that two trajectories starting together will be together at time t. It is absent for $\tilde\lambda=0$. Recall, however, from[11] that in the strongly compressible phase $\wp > \frac{d}{\xi^2}$ one has $\lim\limits_{r\to 0} P^t_{\tilde\lambda}(r;\mathrm{dr}') = \delta(r')\,\mathrm{d}r'$ signaling that the Lagrangian flow is deterministic there. Appearance of both regular and singular contributions to (47) in the sticky flow is one of the indications of a hybrid nature of this case.

In the special instance of $\mu=0$ or $\mu=\infty$, corresponding to instantaneously reflecting or absorbing boundary at $r=0$, the integral in (46) may be calculated explicitly. We make the change of variables $E=z^2$ and then use Weber's formula, Eq. (6.633.2) of ref. 12, to obtain

$$\begin{aligned}\exp(-tN_0)(u,v) &= \int_0^\infty e^{-tE} u^{\frac{1}{2}} J_{-b}\left(\sqrt{E}u\right) v^{\frac{1}{2}} J_{-b}\left(\sqrt{E}v\right) \frac{\mathrm{dE}}{2} \\ &= u^{\frac{1}{2}} v^{\frac{1}{2}} \int_0^\infty e^{-tz^2} J_{-b}(zu) J_{-b}(zv)\, z\, \mathrm{dz} \\ &= \frac{\sqrt{uv}}{2t} \exp\left(-\frac{u^2+v^2}{4t}\right) I_{-b}\left(\frac{uv}{2t}\right)\end{aligned}$$

where I_{-b} is the modified Bessel function of the first kind of index $-b$. Analogously,

$$\begin{aligned}\exp(-tN_\infty)(u,v) &= \int_0^\infty e^{-tE} u^{\frac{1}{2}} J_b(\sqrt{E}u) v^{\frac{1}{2}} J_b(\sqrt{E}v) \frac{\mathrm{dE}}{2} \\ &= \frac{\sqrt{uv}}{2t} \exp\left(-\frac{u^2+v^2}{4t}\right) I_b\left(\frac{uv}{2t}\right).\end{aligned}$$

We then have

$$\begin{aligned}&\exp(tM_{\substack{\infty\\0}})(r,r') \\ &= \frac{1}{\tilde{D}_0(2-\xi)t}\, r^{\frac{1-a_\xi}{2}}\, r'^{\frac{a_\xi+1-2\xi}{2}} \exp\left(-\frac{r^{2-\xi}+r'^{2-\xi}}{\tilde{D}_0(2-\xi)^2 t}\right) I_{\pm b}\left(\frac{2(rr')^{\frac{2-\xi}{2}}}{\tilde{D}_0(2-\xi)^2 t}\right). \quad (48)\end{aligned}$$

It can be easily shown that these results are in agreement with the well known transition probabilities of the Bessel process with reflecting or absorbing boundary at zero.(2)

Some other quantities of interest are the hitting times $H_{r'}$ at some point r', in particular, the doubling and halving times of the inter-particle distance, see e.g. ref. 4, Sect. 2C. Their expectations are given by the following formulae (with notations of Appendix A)

$$\mathbb{E}_r(e^{-\alpha H_{r'}}) = \frac{r^{\frac{\xi-2a_\xi}{4}}\, \phi_{\mu,-\alpha}\left(\frac{2\tilde{D}_0^{-1/2}}{2-\xi} r^{\frac{2-\xi}{2}}\right)}{r'^{\frac{\xi-2a_\xi}{4}}\, \phi_{\mu,-\alpha}\left(\frac{2\tilde{D}_0^{-1/2}}{2-\xi} r'^{\frac{2-\xi}{2}}\right)} \qquad \text{if} \quad r<r'$$

and

$$\mathbb{E}_r(e^{-\alpha H_{r'}}) = \frac{r^{\frac{\xi-2a_\xi}{4}}\, \psi_{-\alpha}\left(\frac{2\tilde{D}_0^{-1/2}}{2-\xi} r^{\frac{2-\xi}{2}}\right)}{r'^{\frac{\xi-2a_\xi}{4}}\, \psi_{-\alpha}\left(\frac{2\tilde{D}_0^{-1/2}}{2-\xi} r'^{\frac{2-\xi}{2}}\right)} \qquad \text{if} \quad r>r'$$

where $\phi_{\mu,-\alpha}$ is the solution of the differential equation $(N+\alpha)\phi=0$ that verifies the boundary condition (42) at zero and $\psi_{-\alpha}$ is the solution that tends to zero at infinity. We refer here to Appendix A and Appendix D, in particular to formula (63) of the former and formulae (68) and (69) of the latter.

5.2. Long Time Asymptotics

The sticky transition probability densities $P^t_{\tilde\lambda}(r;\mathrm{d}r')$ of the inter-particle distance determine the behavior of the passive tracer 2-point function obtained in the corresponding limit $l_\kappa, l_\nu \to 0$. The free decay of the tracer 2-point function is described directly by the transition probability $P^t_{\tilde\lambda}(r;\mathrm{d}r')$. Indeed, for an initial tracer distribution with a homogeneous isotropic 2-point correlation function $F(0,r)$, at time t the tracer distribution will be

$$F(t,r) = \int_0^\infty F(0,r')\, P^t_{\tilde\lambda}(r;\mathrm{d}r'), \tag{49}$$

as follows from (9). Thus the long time decay of the tracer 2-point function is determined by the large t asymptotics of $e^{tM_{\tilde\lambda}}(r,r')$. Similarly, with isotropic forcing as in (10) and, for simplicity, no tracer at time zero,

$$F(t,r) = \int_0^t \mathrm{ds} \int_0^\infty \chi(\mathrm{r}')\, \mathrm{P}^{\mathrm{s}}_{\tilde\lambda}(\mathrm{r};\mathrm{d}\mathrm{r}'), \tag{50}$$

as follows from (11). Thus the long time behavior of the forced tracer 2-point function is determined by the large t asymptotics of $\int_0^t e^{sM_{\tilde\lambda}}(r,r')\,\mathrm{ds}$. We shall assume fast decay for large r of both $F(0,r)$ in the unforced case and of $\chi(r)$ in the forced one.

As before, it will be more convenient to study instead of $M_{\tilde\lambda}$ the operator N_μ. To obtain the large t behavior of $e^{-tN_\mu}(u,v)$ or of $L_\mu(t;u,v) \equiv \int_0^t e^{-sN_\mu}(u,v)\,\mathrm{ds}$, we may consider their Laplace transforms given for $\alpha > 0$ (where α is the variable conjugate to t) by the resolvent kernel

$$(N_\mu + \alpha)^{-1}(u,v) \;=\; \int_0^\infty e^{-\alpha t}\, e^{-tN_\mu}(u,v)\,\mathrm{dt}$$

and by $\frac{1}{\alpha}(N_\mu+\alpha)^{-1}(u,v)$, respectively. By the well known Tauberian-Abelian theorem, for any real numbers $-1 < p_N < \ldots < p_1 < \infty$, the following behaviors of the function $f(t)$ and of its Laplace transform $\hat f(\alpha)$ are equivalent

(i) $$f(t) = \sum_{i=1}^N c_i t^{p_i} + o(t^{p_N}) \qquad \text{near } t=\infty,$$

(ii) $$\hat f(\alpha) = \sum_{i=1}^N c_i \Gamma(p_i+1)\alpha^{-p_i-1} + o(\alpha^{-p_N-1}) \qquad \text{near } \alpha = 0^+ .$$

The resolvent $(N_\mu+\alpha)^{-1}$ of N_μ is studied in Appendix D. It can be written like in (70) (except for the change of sign of α) and expanded in α for α small to all orders ≤ 0. It is easy to see that, for $\mu<\infty$, in the numerator of (70) only the leading term is of order strictly smaller than zero and there is exactly one term of order zero. More exactly, one obtains:

$$\begin{aligned}(N_\mu+\alpha)^{-1}(u,v) &= G_{\mu,-\alpha}(u,v)\\ &= C_b\,(uv)^{\frac{1}{2}-b}\frac{\alpha^{-b}}{1+\mu\alpha^{1-b}}\\ &\quad-\frac{1}{2b}\,\min(u,v)^{\frac{1}{2}-b}\max(u,v)^{\frac{1}{2}+b}+o(\alpha^0)\\ &= C_b\,(uv)^{\frac{1}{2}-b}\sum_{0\leq n\leq\frac{b}{1-b}}(-\mu)^n\,\alpha^{n(1-b)-b}\\ &\quad-\frac{1}{2b}\,\min(u,v)^{\frac{1}{2}-b}\,\max(u,v)^{\frac{1}{2}+b}+o(\alpha^0)\,.\end{aligned}$$

This permits to infer the expansions

$$e^{-tN_\mu}(u,v)=C_b(uv)^{\frac{1}{2}-b}\sum_{0\leq n<\frac{b}{1-b}}\frac{(-\mu)^n}{\Gamma(b-n(1-b))}t^{b-n(1-b)-1}+o(t^{-1+\epsilon})\quad(51)$$

for any $\epsilon>0$, and

$$\begin{aligned}L_\mu(t;u,v) \;=\; & C_b\,(uv)^{\frac{1}{2}-b}\sum_{0\leq n\leq\frac{b}{1-b}}\frac{(-\mu)^n}{\Gamma(1+b-n(1-b))}\,t^{b-n(1-b)}\\ &-\;\frac{1}{2b}\,\min(u,v)^{\frac{1}{2}-b}\,\max(u,v)^{\frac{1}{2}+b}\;+\;o(1)\,.\end{aligned}\qquad(52)$$

They give the large t asymptotics relevant for the study of the long time behavior of the tracer 2-point function in the decaying and forced regime, respectively.

5.3. Consequences for the Tracer Transport

Substituting the expansion (51) into (43) and then into (41), one infers from (49) that the dominant terms in the free decay of the tracer 2-point function are r-independent:

$$F(t,r) \;=\; (\tfrac{2-\xi}{2})^{2b-1}\tilde{D}_0^{b-1}C_b \sum_{0\le n<\frac{b}{1-b}} \frac{(-\mu)^n}{\Gamma(b-n(1-b))}\,t^{b-n(1-b)-1} \cdot\Big(\int_0^\infty r'^{\,a-\xi}\,F(0,r')\,\mathrm{d}r' + \tilde{\lambda}\,\mathrm{F}(0,0)\Big) \;+\; \mathrm{o}(\mathrm{t}^{-1+\epsilon})\,.$$

Similarly, from (52), (43), (41) and, finally, (50), we infer that in the forced situation

$$F(t,r) = (\tfrac{2-\xi}{2})^{2b-1}\tilde{D}_0^{b-1}C_b \sum_{0\le n\le\frac{b}{1-b}} \frac{(-\mu)^n}{\Gamma(1+b-n(1-b))}\,t^{b-n(1-b)}\Big(\int_0^\infty r'^{\,a-\xi}\,\chi(r')\,\mathrm{d}r' + \tilde{\lambda}\,\chi(0)\Big) - \tfrac{1}{\tilde{D}_0(1-a)}\Big(r^{1-a}\int_0^r r'^{a-\xi}\,\chi(r')\,\mathrm{d}r' + \int_{\mathrm{r}}^\infty \mathrm{r}'^{1-\xi}\chi(\mathrm{r}')\,\mathrm{d}\mathrm{r}' + \tilde{\lambda}\,\mathrm{r}^{1-\mathrm{a}}\,\chi(0)\Big) + \mathrm{o}(1)\,. \quad (53)$$

It is interesting to notice that the number of terms appearing in this expansion is variable depending on the value of the integer part of $\frac{b}{1-b}$. In particular if $n(1-b)-b=0$, i.e. $b=1-\frac{1}{n+1}$ for some natural number n, then there is a supplementary term of order 0 in the forced 2-point function.

For $\tilde{\lambda}=0$, the result (53) reduces to the one worked out in ref. 11. The physics of the solutions with $0<\tilde{\lambda}<\infty$ is not very different from the one for $\tilde{\lambda}=0$ described in Sect. 3.2 and 3.3 of ref. 11. The tracer "energy" with density $\frac{1}{2}\theta^2$ is dissipated but, in the forced case, also building up in the constant mode growing like t^b. Non-zero $\tilde{\lambda}$ brings subleading terms in this buildup proportional to $t^{b-n(1-b)}$. In the stationary state the mean tracer energy density $\langle\frac{1}{2}\theta^2\rangle$ is infinite. The rate of pumping of the constant mode $\propto t^{b-1}$ goes, however, to zero. The tracer energy dissipation rate equal to $-\frac{1}{2}MF(t)|_{r=0}$ approaches at long times the stationary value equal to the injection rate $\frac{1}{2}\chi(0)$ and the stationary state exhibits a *direct energy cascade*. The tracer 2-point structure function approaches the stationary form

$$\begin{aligned} S_2(t,r) &= 2(F(t,0)-F(t,r)) \\ \underset{t\to\infty}{\longrightarrow} &\quad \tfrac{2}{\tilde{D}_0}\Big[\int_{0<r'<r''<r} r''^{\,-a_\xi}\,r'^{\,a_\xi-\xi}\,\chi(r')\,\mathrm{d}r'\,\mathrm{d}r'' + \tfrac{1}{1-\mathrm{a}_\xi}\,\tilde{\lambda}\,\mathrm{r}^{1-\mathrm{a}_\xi}\,\chi(0)\Big]. \end{aligned} \quad (54)$$

For small r the integral on the right hand side scales as $r^{2-\xi}$ while the term multiplying $\tilde{\lambda}$ is proportional to r^{1-a_ξ}. Since $1-a_\xi<2-\xi$, a non-zero $\tilde{\lambda}$ has an important effect. It changes the normal small r scaling $\propto r^{2-\xi}$ of the 2-point structure function occurring for $\tilde{\lambda}=0$ to the *anomalous scaling* proportional to the zero mode r^{1-a_ξ} of M.

Let us also briefly mention the case $\tilde{\lambda}=\infty$, i.e. the one with absorbing boundary condition and coalescence of trajectories that was not covered above (the limits $t\to\infty$ and $\tilde{\lambda}\to\infty$ do not commute!). Using (48), the transition probabilities take in this case the form

$$P^t_\infty(r;dr') \;=\; e^{tM_\infty}(r,r')\,dr' + \Big[1-\gamma(b,\tfrac{r^{2-\xi}}{(2-\xi)^2\tilde{D}_0\,t})\,\Gamma(b)^{-1}\Big]\,\delta(r')\,dr'\,,$$

the same as in the strongly compressible phase, see Eq. (2.26) of ref. 11. The coefficient of $\delta(r')$ is obtained simply by calculating the missing mass of $e^{tM_\infty}(r,r')$. For large time t, the 2-point function of the forced tracer becomes:

$$F(t,r) = \Big(t-\tfrac{1}{(2-\xi)^{2b}(1-b)\,\Gamma(1+b)}\tilde{D}_0^{-b}\,t^{1-b}\,r^{1-a}+\tfrac{1}{(2-\xi)^2(1-b)}\tilde{D}_0^{-1}\,r^{2-\xi}\Big)\chi(0)$$
$$+\tfrac{1}{(1-a)\tilde{D}_0}\Big(\int_0^r r'^{\,1-\xi}\,\chi(r')\,dr'+r^{1-a}\int_r^\infty r'^{\,a-\xi}\,\chi(r')\,dr'\Big)+\;o(1)\,.$$

The behavior of this solution is similar to that of the strongly compressible phase analyzed in Sect. 3.4 of ref. 11 and quite different from the one for finite $\tilde{\lambda}$. The scalar energy builds up linearly in time in the constant mode in an *inverse cascade* process and there is no persistent dissipation. At difference with the solution in the strongly compressible phase, however, the tracer 2-point structure function does not reach a stationary regime due to the contribution proportional to the zero mode r^{1-a} growing in time like t^{1-b}.

6. EXACT LIMIT OF THE STATIONARY 2-POINT TRACER STRUCTURE FUNCTION

In the main part of the paper we have used in the calculations the approximate forms of functions f_1 and f_2 instead of their exact versions. The question arises whether such calculations reproduce the actual behavior of the limiting dispersion process, up to finite renormalization of the glue parameter $\tilde{\lambda}$. Besides, one would like to have exact expressions for some quantities, like the mass at zero of the limiting speed measure or the limiting stationary 2-point structure function of the tracer. It turns out that it is indeed possible to obtain precise formulae and that the asymptotic behavior of the integrals that we calculated approximately differs from the exact one by a (finite, non-zero) multiplicative renormalization.

In what follows, we shall need detailed information about functions $f_{1,2}(r;l_\nu,l_\kappa)$ and their ratio $f_2(r;l_\nu,l_\kappa)/f_1(r;l_\nu,l_\kappa)$ that is described in

Appendix F. In particular we prove there the positivity for $r > 0$ and the scaling properties

$$f_1(r; l_\nu) = l_\nu^\xi \, f_1(\tfrac{r}{l_\nu}; 1), \qquad f_2(r; l_\nu) = l_\nu^{\xi-1} \, f_1(\tfrac{r}{l_\nu}; 1) \tag{55}$$

of functions $f_{1,2}(r; l_\nu) \equiv f_{1,2}(r; l_\nu, 0)$. We also establish two decompositions. The first one:

$$f_1(\rho; 1) = \tilde{C}_0 \, \rho^2 f_1^{\clubsuit}(\rho) = \tilde{C}_0 \, \rho^2 + \rho^4 f_1^{\spadesuit}(\rho) \tag{56}$$

$$f_2(\rho; 1) = a_2 \, \tilde{C}_0 \, \rho f_2^{\clubsuit}(\rho) = a_2 \, \tilde{C}_0 \, \rho + \rho^3 f_2^{\spadesuit}(\rho) \tag{57}$$

with $\lim\limits_{\rho \to 0} f_{1,2}^{\clubsuit}(\rho) = 1$ will be used for small ρ. The second one:

$$f_1(\rho; 1) = \rho^\xi f_1^{\diamondsuit}(\rho) = \rho^\xi + \rho^{\xi-2} f_1^{\heartsuit}(\rho) \tag{58}$$

$$f_2(\rho; 1) = a_\xi \, \rho^{\xi-1} f_2^{\diamondsuit}(\rho) = a_\xi \, \rho^{\xi-1} + \rho^{\xi-3} f_2^{\heartsuit}(\rho) \tag{59}$$

with $\lim\limits_{\rho \to \infty} f_i^{\diamondsuit}(\rho) = 1$ will be employed for large ρ. The coefficients a_2 and a_ξ are given by (18). The above decompositions are used in Appendix F to establish bounds (83) to (85) on the ratio $f_2(r; l_\nu, l_\kappa)/f_1(r; l_\nu, l_\kappa)$. Given such bounds, it is immediate to see that there exists a constant B' such that, for any $r > 0$ and any $0 < l_\kappa < l_\nu < 1$, the density $s(r)$ of the natural scale measure given by (22) is comprised between $1/B'$ times and B' times the approximate expressions listed in Table III. Using additionally estimates (81) to (82) from Appendix F, one infers that the same statement holds for the density $m(r) = [f_1(r) s(r)]^{-1}$ of the speed measure. It is now easy to show the convergence of $s(r)$ and $m(r)$ to $s_0(r)$ and $m_0(r)$ given by (24) and (30) for $r > 0$. Controlling what happens around $r = 0$ will be the main difficulty.

After the above preparation, we pass to the main topic of this section: the proof of convergence, when $l_\nu, l_\kappa \to 0$, of the stationary 2-point structure function of the forced passively advected tracer. We shall show that, under the condition (29) with $\lambda < \infty$, the limit exists and corresponds to the structure function computed directly at $l_\nu, l_\kappa = 0$ with the sticky boundary condition at zero, see (54). Besides, we shall establish a precise relation between λ and the glue parameter $\tilde{\lambda}$ of the boundary condition (37).

For l_ν, l_κ positive, the stationary 2-point structure function is the unique solution vanishing at zero together with its first derivative of the equation $M_{\nu,\kappa} S_2 = 2\chi$. It is given by the relation:

$$S_2(r) = \frac{2}{\tilde{D}_0} \int_0^r s(r'') \, \mathrm{d}r'' \int_0^{r''} \mathrm{m}(r') \, \chi(r') \, \mathrm{d}r' = \frac{2}{\tilde{D}_0} \int\limits_{0<r'<r''<r} \mathrm{s}(r'') \, \mathrm{m}(r') \, \chi(r') \, \mathrm{d}r' \, \mathrm{d}r'' .$$

The last integral may be cut up into six pieces according to the positions of r', r'' with respect to l_κ, l_ν. The bounds given for $s(r)$ and $m(r)$ indicate that in each domain the dominant behavior of the contribution to $S_2(r)$ is estimated correctly by using the rough forms in Table III, up to multiplication by a constant bounded independently of l_κ, l_ν and r. We get the following behaviors. The domain $l_\nu < r' < r''$ gives a contribution behaving at lowest order like $r^{2-\xi}$ for small r and like r^{1-a_ξ} for large r. Domains $l_\kappa < r' < l_\nu < r''$ and $r' < l_\kappa < l_\nu < r''$ both give terms of order $l_\nu^{2-\xi+a_\xi-a_2} l_\kappa^{a_2-1} r^{1-a_\xi}$. Finally the contribution from domains $r'' < r' < l_\nu$ is always subdominant compared to the former ones. This suggests that the limiting value of $S_2(r; l_\nu, l_\kappa)$ will depend on the limit of $l_\nu^{2-\xi+a_\xi-a_2} l_\kappa^{a_2-1}$ as l_ν, l_κ go to zero. The only task left is to calculate the limit of $S_2(r; l_\nu, l_\kappa)$ for $l_\nu^{2-\xi+a_\xi-a_2} l_\kappa^{a_2-1} \to \lambda$ with $\lambda < \infty$.

We already know that the domains with $0 < r'' < r' < l_\nu$ will give vanishing contributions (supposing that the others give a finite one). Let us turn to the other three. The easiest to handle is $l_\nu < r'' < r' < r$. Because of the bounds on $s(r)$ and $m(r)$ we can immediately use the Dominated Convergence Theorem to obtain

$$\begin{aligned}
\lim_{l_\nu, l_\kappa \to 0} & \int\limits_{l_\nu < r' < r'' < r} s(r''; l_\nu, l_\kappa)\, m(r'; l_\nu, l_\kappa)\, \chi(r')\, \mathrm{d}r'\, \mathrm{d}r'' \\
&= \int\limits_{0 < r' < r'' < r} s(r''; 0, 0)\, m(r'; 0, 0)\, \chi(r')\, \mathrm{d}r'\, \mathrm{d}r'' \\
&= \int\limits_{0 < r' < r'' < r} r''^{\,-a_\xi}\, r'^{\,a_\xi - \xi}\, \chi(r')\, \mathrm{d}r'\, \mathrm{d}r''.
\end{aligned}$$

Finally we turn to the two remaining domains covering the region $0 < r' < l_\nu < r'' < r$. The corresponding integral factorizes:

$$\begin{aligned}
& \int\limits_{0 < r' < l_\nu < r'' < r} s(r''; l_\nu, l_\kappa)\, m(r'; l_\nu, l_\kappa)\, \chi(r')\, \mathrm{d}r'\, \mathrm{d}r'' \\
&= \left[\int_{l_\nu}^{r} s(r''; l_\nu, l_\kappa)\, \mathrm{d}r''\right] \left[\int_0^{l_\nu} m(r'; l_\nu, l_\kappa)\, \chi(r')\, \mathrm{d}r'\right].
\end{aligned}$$

To the first factor, one may again apply the Dominated Convergence Theorem to obtain

$$\lim_{l_\nu, l_\kappa \to 0} \int_{l_\nu}^{r} s(r''; l_\nu, l_\kappa)\, dr'' = \int_0^r s(r''; 0, 0)\, dr'' = \int_0^{\mathrm{r}} \left(\frac{\mathrm{r}_0}{\mathrm{r}''}\right)^{\mathrm{a}_\xi} dr'' = \frac{1}{1-\mathrm{a}_\xi}\, \mathrm{r}_0^{\mathrm{a}_\xi}\, \mathrm{r}^{1-\mathrm{a}_\xi}.$$

Control of the second factor, that converges to the mass at zero of the limiting speed measure multiplied by $\chi(0)$, is the crucial element of the argument and it requires technical work. We postpone it to Appendix G. We prove there that for $\lambda < \infty$,

$$\lim_{l_\nu, l_\kappa \overset{\lambda}{\to} 0} \int_0^{l_\nu} m(r'; l_\nu, l_\kappa)\, \chi(r')\, dr' = \mathrm{r}_0^{-\mathrm{a}_\xi}\, \mathrm{Y}\, \lambda\, \chi(0) \tag{60}$$

with

$$Y = \frac{1}{4}\left(\frac{1}{2}\tilde{C}_0\right)^{-\frac{a_2+1}{2}} B\left(\frac{d}{2}, \frac{1-a_2}{2}\right) \cdot \exp\left[-\int_0^1 \frac{\rho\left[f_2^{\spadesuit}(\rho) - a_2 f_1^{\spadesuit}(\rho)\right]}{\tilde{C}_0 f_1^{\clubsuit}(\rho)}\, d\rho - \int_1^\infty \frac{\mathrm{f}_2^{\heartsuit}(\rho) - \mathrm{a}_\xi \mathrm{f}_1^{\heartsuit}(\rho)}{\rho^3 \mathrm{f}_1^{\diamondsuit}(\rho)}\, d\rho\right] \tag{61}$$

where $l_\nu, l_\kappa \overset{\lambda}{\to} 0$ denotes the limit $l_\nu, l_\kappa \to 0$ under condition (29) and $B(x, y)$ is the Euler Beta function. The constant Y is positive. The mass at zero of the limiting speed measure is then equal to $r_0^{-a_\xi} Y \lambda$ which should be contrasted with the approximate formula (31). Given the above result, the contribution of the region $0 < r' < l_\nu < r'' < r$ to $S_2(r)$ becomes

$$\frac{2Y}{\tilde{D}_0(1-a_\xi)}\, \lambda\, r^{1-a_\xi}\, \chi(0).$$

The final outcome is the relation:

$$\lim_{l_\nu, l_\kappa \overset{\lambda}{\to} 0} S_2(r) = \frac{2}{\tilde{D}_0}\Big(\int\limits_{0<r'<r''<r} r''^{\,-a_\xi} r'^{\,a_\xi - \xi}\, \chi(r')\, dr'\, dr'' + \frac{\mathrm{Y}}{1-\mathrm{a}_\xi}\, \lambda\, \mathrm{r}^{1-\mathrm{a}_\xi}\, \chi(0)\Big)$$

which agrees with expression (54) of Section 5.3 if we put $\tilde{\lambda} = Y\lambda$. This means that the structure function obtained in the $l_\nu, l_\kappa \overset{\lambda}{\to} 0$ limit coincides with the one that is obtained directly using the sticky process with the generator $M_{\tilde{\lambda}}$ for the value

$$\tilde{\lambda} = Y\lambda \tag{62}$$

of the glue parameter. Note that the effect of using the exact versions of f_1, f_2 instead of the approximate versions of table Table II manifests itself only in the change of the proportionality constant between λ and the glue parameter $\tilde{\lambda}$, i.e. in a finite multiplicative renormalization of $\tilde{\lambda}$.

In view of all that has been said, it is natural to expect that the entire Lagrangian pair dispersion process behaves in the $l_\nu, l_\kappa \stackrel{\lambda}{\to} 0$ limit as the diffusion process with the generator $M_{\tilde{\lambda}}$ for the glue parameter $\tilde{\lambda}$ given by (62). In particular, such a process is equivalent in the coordinate (19) to the Bessel process of parameter $-b_{\xi,\wp}$ with a sticky behavior at zero.

7. CONCLUSIONS

We have analyzed in this article the small viscosity, small diffusivity behavior of the Lagrangian dispersion in the Kraichnan model with intermediate compressibility degree $\frac{d-2}{2\xi} + \frac{1}{2} < \wp < \frac{d}{\xi^2}$. In this interval, the Lagrangian trajectories may separate fast due to the spatial roughness of velocities but may also come close due to the trapping effects of compressibility. As first suggested in refs. 5 and 6, we have discovered different possible asymptotic regimes of the Lagrangian flow, depending on the limiting behavior of the Prandtl number when the viscous and diffusive cut-off scales l_ν and l_κ are taken to zero. This arbitrariness reflects the frustration of the particles unable to choose between opposite trends of life. Specifically, we have argued that there are different limits of the dispersion process depending on the behavior of the combination

$$l_\nu^{a_\xi+2-\xi-a_2}\, l_\kappa^{a_2-1} \;=\; \text{const.}\; l_\nu^{a_\xi+1-\xi}\,(Pr)^{\frac{1-a_\xi}{2}}$$

of the cutoff scales. If this combination goes to zero when l_ν and l_κ are sent to zero, the resulting dispersion process is that of trajectories instantaneously reflecting off each other upon hitting. If it goes to infinity, the trajectories coalesce when they meet, behaving similarly as in the strongly compressible regime. Finally, if the above combination goes to a finite limit λ when $l_\nu, l_\kappa \to 0$ (which sends to infinity the Prandtl number at a specific pace) then the resulting dispersion process exhibits the sticky or slowly reflecting behavior with λ proportional to the amount of "glue" keeping the particles together. Such a behavior leaves a visible imprint on the passive advection of tracers in the subleading contributions to the tracer energy condensation in the zero wave number and, even more dramatically, by generating anomalous scaling of the stationary 2-point structure function of the tracer. The analysis in the main part of the paper was based on approximate calculations of the asymptotic behavior of the natural scale and speed measure of the Lagrangian dispersion process and of the eigenfunctions of its generator. To set the results on a firmer ground we have also shown rigorously that the finite l_ν, l_κ stationary 2-point structure function of the tracer converges, under the limit $l_\nu, l_\kappa \to 0$ with finite λ, to the stationary structure function obtained directly from the sticky dispersion process. This argument allowed to fix exactly the proportionality constant between λ and the value of the glue parameter. Undoubtedly, with a little more work controlling the convergence of resolvents of $M_{\nu,\kappa}$ to that of $M_{\tilde{\lambda}}$ one should be able to prove, along the lines of ref. 9 that the laws of the dispersion processes for positive l_ν, l_κ converge to the law of the sticky process.

The main open problem, untouched by our analysis, is the construction of N-particle processes corresponding to the sticky behavior of the two-particle dispersion. In particular it would be interesting to know whether the amount of two-particle glue is the only parameter that labels possible Lagrangian flows in the moderately compressible phase of the Kraichnan model. The Dirichlet form approach used in ref. 18 in the 1-dimensional $\xi = 0$ case to tackle such questions is unavailable in the other instances, at least in its classical form, due to the lack of symmetry of the generators of the N-particle processes. Further open questions of fundamental importance concern possible occurrence of sticky Lagrangian flows in more realistic velocity ensembles exhibiting fully developed turbulence.

APPENDIX A. BRIEFLY ON ONE-DIMENSIONAL DIFFUSION PROCESSES

We collect here some facts, used in the main text, about one-dimensional diffusion processes. The reader may wish to consult the relevant literature (e.g. refs. 3 and 22) for an extensive treatment.

Let $X(t)$ be a stochastic diffusion process on $\mathbb{R}_+$. The Kolmogorov backwards evolution operator P^t of the process, acting on bounded continuous functions f defined on $\mathbb{R}_+$, is given by

$$(P^t f)(r) = \mathbb{E}_r f(X(t)) = \int f(r') P^t(r, \mathrm{d}r')$$

where we define $\mathbb{E}_r f(X(t)) \equiv \mathbb{E}(f(X(t)) \,|\, X(0) = r)$. The measures $P^t(r, \mathrm{d}r')$ giving the kernels of operators P^t are the transition probabilities of the process. The family $(P^t)_{t \in \mathbb{R}_+}$ forms a one-parameter semigroup whose generator $\partial_t P^t|_{t=0}$ we shall denote by M. In general M is a second order differential operator. The transition probabilities verify the PDE

$$\partial_t P^t(r, \mathrm{d}r') = \mathrm{M(r)P^t(r, dr')}\,.$$

Any regular diffusion process may be derived from Brownian motion by an adequate change of variables defined in terms of the natural scale and the speed measure associated to the process. The natural scale is defined as the unique, up to affine transformations, continuous strictly increasing function S such that $S(X(t))$, i.e. $X(t)$ considered in the new spatial coordinate S, is a martingale. S is called the natural scale and $s(r)=\frac{\mathrm{dS(r)}}{\mathrm{dr}}$ the density of the natural scale measure with respect to the coordinate r on $\mathbb{R}_+$.

If the range of S is the whole real line then the process $S(X(t))$, for $X(0)$ fixed, has the same distribution as an appropriately time-changed Brownian motion starting from $S(X(0))$. This means that if W is a Brownian motion starting from $S(X(0))$, then $S(X(t))$ has the same law as $W(\tau(t))$ for some function $\tau(t)=\tau(t;W)$ which depends also on the realization of the Brownian path. Let us denote by $L(\tau,S;W)$ the local time of the Brownian path W at the point S up to instant τ. Formally, $L(\tau,S;W)=\int_0^\tau \delta(S-W(\sigma))\,\mathrm{d}\sigma$. The speed measure of the process $X(t)$ is defined as the unique positive measure dm(S) on $\mathbb{R}$ such that the relation between t and τ may be written as

$$\tau(t;W)=\inf\left\{\sigma:\int L(\sigma,S;W)\,\mathrm{dm(S)}>\mathrm{t}\right\}$$

or, equivalently,

$$t(\tau;W)=\int L(\tau,S;W)\,\mathrm{dm(S)}\,.$$

Intuitively, the speed measure expresses how much time the (time-changed) Brownian motion needs to advance "one step" at a given point in space. The larger the (density of the) speed measure at some point, the slower the (time-changed) Brownian motion advances.

The generator M of the Kolmogorov backward evolution semigroup may be written in terms of the densities $m(S)=\frac{\mathrm{dm(S)}}{\mathrm{dS}}$ or $m(r)=\frac{\mathrm{dm(r)}}{\mathrm{dr}}$ and $s(r)$ as

$$M=\frac{1}{2m(S)}\partial_S^2=\frac{1}{2m(r)}\partial_r\circ\frac{1}{s(r)}\partial_r\,.$$

Similarly, if the range of S is a positive half-line, which may be chosen as $\mathbb{R}_+$, then analogous statements hold with the Brownian motion W replaced by the Brownian motion reflecting at zero, i.e. by $|W(t)|$. In any case, the behavior of the process at the boundary point $r=0$ may be classified according to Feller's criteria expressible in terms of the natural scale and the speed measure. When $r=0$ is a regular boundary point, the behavior of the process at this point depends on the mass $m(\{0\})$ w.r.t the speed measure and is reflected in the boundary condition for the generator M.

Finally let us recall from ref. 13 the following formula for hitting probabilities of some process X_t. Denote by $H_{r'}$ the hitting time of the process at some point r'. Then for $\alpha > 0$

$$\mathbb{E}_r(e^{-\alpha H_{r'}}) = \frac{\phi(r)}{\phi(r')} \tag{63}$$

where ϕ is the solution of $(M-\alpha)\phi = 0$ that verifies the correct boundary condition either on the left (i.e. at 0 if the process is defined on the half-line) if $r < r'$ or on the right (i.e. at infinity if the process is defined on the half-line) if $r > r'$.

APPENDIX B. A PATHOLOGICAL CASE OF CONVERGENCE

Let L be an arbitrary length and l a small scale that we shall send to zero. A diffusion process may be specified by giving its natural scale and its speed measure. Consider such a process on $\mathbb{R}_+$ whose natural scale is

$$S(r) = \begin{cases} l - L + \frac{L}{l} r & \text{for} \quad r \in [0, l], \\ r & \text{for} \quad r \in [l, \infty) \end{cases}$$

so that the density of the natural scale measure w.r.t. r is

$$s(r) = \begin{cases} \frac{L}{l} & \text{for} \quad r \in [0, l], \\ 1 & \text{for} \quad r \in [l, \infty). \end{cases}$$

Let us take for (the density w.r.t. r of) the speed measure

$$m(r) = \begin{cases} \frac{L}{l} & \text{for} \quad r \in [0, l], \\ 1 & \text{for} \quad r \in [l, \infty). \end{cases}$$

We see that taking the limit $l \to 0$ for $r > 0$ we get $s_0(r) = 1$ corresponding to $S_0(r) = r$, however $\lim_{l \to 0} S(0) = -L$. This is not very different from the case studied in Section 3, where $S(0) = -\infty$ as long as there is finite regularization. For the limit of the speed measure we get $m_0(r) = L\delta(r) + 1$.

Let us now pass to the natural scale first. Then $m(S) = \frac{m(r)}{s(r)} = 1$ for all S. Thus $m_0(S) = 1$. This is of course incompatible with the previous result.

The explanation of this phenomenon is the following. To the right of l the process is just Brownian motion. To the left of l it is Brownian motion on a segment of length L (with reflecting left end) "squeezed" into the segment $[0,l]$. Thus when we take l to zero, the limiting process will be a Brownian motion on $\mathbb{R}_+$ which "sees" an additional segment of length L to its left. This is of course a non-Markovian boundary condition.

APPENDIX C. ASYMPTOTIC BEHAVIOR OF EIGENFUNCTIONS

Starting from formula (34) we calculate the limit of c_E^+/c_E^- when $l_\kappa, l_\nu \to 0$. We assume that $l_\kappa < l_\nu$, but not necessarily $l_\kappa \ll l_\nu$. The calculation can be done in the usual way by expanding c_E^+/c_E^- in a multivariate power series. First we expand every term in (34) to the *minimal orders* in l_ν, l_κ. That is, we expand the expressions into a power series (and possibly powers of logarithms) in l_ν, l_κ and keep all the terms such that there is no term of smaller order simultaneously in l_ν and in l_κ.

We list below the expansion to minimal orders in l_κ, l_ν of all terms appearing in (34):

$$g_1^-(l_\kappa)\sim 1 \qquad g_1^{-\prime}(l_\kappa)\sim -\frac{E}{d}l_\nu^{2-\xi}l_\kappa^{-1}$$

$$\gamma^-\sim\frac{E}{1-a_2}l_\nu^{2-\xi} \qquad \gamma^+\sim 1-a_2$$

$$g_2^-(l_\kappa)=(l_\kappa)^{\gamma^-} \qquad g_2^{-\prime}(l_\kappa)=\gamma^-(l_\kappa)^{\gamma^- -1}\sim\frac{E}{1-a_2}l_\nu^{2-\xi}l_\kappa^{-1}(l_\kappa)^{\gamma^-}$$

$$g_2^+(l_\kappa)=(l_\kappa)^{\gamma^+}\sim l_\kappa^{1-a_2} \qquad g_2^{+\prime}(l_\kappa)=\gamma^+(l_\kappa)^{\gamma^+-1}\sim(1-a_2)l_\kappa^{-a_2}$$

$$g_2^-(l_\nu)=(l_\nu)^{\gamma^-}\sim e^{\frac{E}{1-a_2}l_\nu^{2-\xi}\ln l_\nu}\sim 1 \qquad g_2^{-\prime}(l_\nu)=\gamma^-(l_\nu)^{\gamma^- -1}\sim\frac{E}{1-a_2}l_\nu^{1-\xi}$$

$$g_2^+(l_\nu)=(l_\nu)^{\gamma^+}\sim l_\nu^{1-a_2} \qquad g_2^{+\prime}(l_\nu)=\gamma^+(l_\nu)^{\gamma^+-1}\sim(1-a_2)l_\nu^{-a_2}$$

$$g_3^-(l_\nu)\sim 1 \qquad g_3^{-\prime}(l_\nu)\sim-\frac{(2-\xi)^2E}{4(1-\xi+a_\xi)}l_\nu^{1-\xi}$$

$$g_3^+(l_\nu)\sim l_\nu^{1-a_\xi} \qquad g_3^{+\prime}(l_\nu)\sim(1-a_\xi)l_\nu^{-a_\xi}$$

Note that $(l_\kappa)^{\gamma^-}$ cannot be further expanded without any additional hypothesis on the relative behaviors of l_ν and l_κ because both the base and the exponent go to zero and one depends on l_κ the other on l_ν. All we can say is that $l_\kappa^\epsilon=o((l_\kappa)^{\gamma^-})$ for any $\epsilon>0$, and $(l_\kappa)^{\gamma^-}<1$ when $l_\kappa<1$. We shall keep $(l_\kappa)^{\gamma^-}$ as it is in the expansions.

Calculations to minimal order with expansions into multivariate power series are a little trickier then with univariate expansions. Even if the coefficients of some but not all monomials in the expansion simplify to zero at intermediate stages then possibly higher order terms should be taken into account because they could give rise to terms of minimal order.[6] This precaution is implicit in the computations. However, as long as there is only one term of minimal order (and it does not simplify out), no special care is needed.

After having determined above the behavior of each term, we may now calculate the sub-products of (34) to minimal order. First evaluate

$$\begin{pmatrix} g_3^{-\prime}(l_\nu) & -g_3^-(l_\nu) \\ -g_3^{+\prime}(l_\nu) & g_3^+(l_\nu) \end{pmatrix} \begin{pmatrix} g_2^+(l_\nu) & g_2^-(l_\nu) \\ g_2^{+\prime}(l_\nu) & g_2^{-\prime}(l_\nu) \end{pmatrix} \sim \begin{pmatrix} -(1-a_2)\, l_\nu^{-a_2} & -\left[\frac{(2-\xi)^2}{4(1-\xi+a_\xi)} + \frac{1}{1-a_2}\right] E\, l_\nu^{1-\xi} \\ (a_\xi - a_2)\, l_\nu^{1-a_2-a_\xi} & -(1-a_\xi)\, l_\nu^{-a_\xi} \end{pmatrix}.$$

It is straightforward to check that every coefficient is different from zero, except for $a_\xi - a_2$ which may be equal to zero. So there the expansion has to be pushed further, but since each following term is of higher order both in l_ν and in l_κ, we may simply replace $(a_\xi - a_2)\, l_\nu^{1-a_2-a_\xi}$ by $O(l_\nu^{1-a_2-a_\xi})$. Next evaluate

$$\begin{pmatrix} g_2^{-\prime}(l_\kappa) & -g_2^-(l_\kappa) \\ -g_2^{+\prime}(l_\kappa) & g_2^+(l_\kappa) \end{pmatrix} \begin{pmatrix} g_1^-(l_\kappa) \\ g_1^{-\prime}(l_\kappa) \end{pmatrix} \sim \begin{pmatrix} \left[\frac{1}{1-a_2} + \frac{1}{d}\right] E\, l_\nu^{2-\xi} l_\kappa^{-1} (l_\kappa)^{\gamma^-} \\ -(1-a_2)\, l_\kappa^{-a_2} \end{pmatrix}.$$

Again, it is straightforward to check that every coefficient is different from zero. We may multiply together (exactly) the preceding two subexpressions to arrive at

$$\begin{pmatrix} c_E^+ \\ c_E^- \end{pmatrix} \underset{\sim}{\propto} \begin{pmatrix} -\frac{d+1-a_2}{d} E\, l_\nu^{2-\xi-a_2} l_\kappa^{-1} (l_\kappa)^{\gamma^-} + \left[1 + \frac{(2-\xi)^2(1-a_2)}{4(1-\xi+a_\xi)}\right] E\, l_\nu^{1-\xi} l_\kappa^{-a_2} \\ O\left(l_\nu^{3-a_2-a_\xi-\xi} l_\kappa^{-1} (l_\kappa)^{\gamma^-}\right) + (1-a_2)(1-a_\xi)\, l_\nu^{-a_\xi} l_\kappa^{-a_2} \end{pmatrix}. \quad (64)$$

Now all the terms are of minimal order. Additionally, we have

$$l_\nu^{3-a_2-a_\xi-\xi} l_\kappa^{-1} (l_\kappa)^{\gamma^-} = l_\nu^{1-a_\xi} (l_\nu^{2-\xi-a_2} l_\kappa^{-1} (l_\kappa)^{\gamma^-}) = o(l_\nu^{2-\xi-a_2} l_\kappa^{-1} (l_\kappa)^{\gamma^-}), \quad (65)$$

[6]Take for example the sum of two polynomials $(X + XY + Y^2) + (-X + Y^2)$. The sum of the minimal order expansions is $(X + Y^2) + (-X + Y^2) = 2Y^2$ but the minimal order expansion of the sum is of course $XY + 2Y^2$.

$$l_\nu^{1-\xi} l_\kappa^{-a_2} = l_\nu^{a_\xi+1-\xi}(l_\nu^{-a_\xi} l_\kappa^{-a_2}) = o(l_\nu^{-a_\xi} l_\kappa^{-a_2})\,. \tag{66}$$

Let us first suppose that $l_\nu^{2-\xi-a_2} l_\kappa^{-1}(l_\kappa)^{\gamma^-} = o(l_\nu^{-a_\xi} l_\kappa^{-a_2})$. In this case c_E^+/c_E^- goes to zero. This is so even if in (64) the expansion for c_E^+ cancels out, meaning that the expansion should be pushed further. Indeed, subsequent terms in the expansion are of higher order thus asymptotically smaller. On the other hand, there can be no cancellation in the expansion of c_E^- because (65) and the hypothesis of the present paragraph combine to $l_\nu^{3-a_2-a_\xi-\xi} l_\kappa^{-1}(l_\kappa)^{\gamma^-} = o(l_\nu^{-a_\xi} l_\kappa^{-a_2})$.

Conversely, if we suppose $l_\nu^{-a_\xi} l_\kappa^{-a_2} = o(l_\nu^{2-\xi-a_2} l_\kappa^{-1}(l_\kappa)^{\gamma^-})$, then c_E^+/c_E^- goes to ∞. This even if in (64) the expansion for c_E^- cancels out, meaning that the expansion should be pushed further. Once again, subsequent terms in the development are of higher order thus asymptotically smaller. And now it is in the development of c_E^+ that there can be no cancellation because (66) and the hypothesis of the present paragraph combine to $l_\nu^{1-\xi} l_\kappa^{-a_2} = o(l_\nu^{2-\xi-a_2} l_\kappa^{-1}(l_\kappa)^{\gamma^-})$.

In consequence, if c_E^+/c_E^- should have a finite non-zero limit, then $l_\nu^{2-\xi-a_2} l_\kappa^{-1}(l_\kappa)^{\gamma^-}$ and $l_\nu^{-a_\xi} l_\kappa^{-a_2}$ must be of the same order. In particular they are much bigger than the other two terms so there are no cancellations in (64). Also it is easy to see that in this case l_κ cannot decrease faster than some power of l_ν. If we write $l_\nu^{-a_\xi} l_\kappa^{-a_2} = O(l_\nu^{2-\xi-a_2} l_\kappa^{-1}(l_\kappa)^{\gamma^-})$ then, using $(l_\kappa)^{\gamma^-} = O(1)$, we have $l_\nu^{\frac{2-\xi-a_2+a_\xi}{1-a_2}} = O(l_\kappa)$. This implies $(l_\kappa)^{\gamma^-} \sim 1$. We may then conclude that

$$\frac{c_E^+}{c_E^-} \sim -\frac{d+1-a_2}{(1-a_2)(1-a_\xi)d}\, E\, l_\nu^{2-\xi-a_2+a_\xi} l_\kappa^{a_2-1}\,.$$

This expression has a finite limit if and only if $l_\nu^{2-\xi-a_2+a_\xi} l_\kappa^{a_2-1}$ goes to some finite limit λ, just as in (29). We obtain this way formula (35). In fact our proof shows that (35) is valid also if λ is zero or infinite.

APPENDIX D. SPECTRAL ANALYSIS OF N_μ

The (operator valued) spectral measure for the operator N_μ may be evaluated with the help of the formula

$$E_\mu(\mathcal{B}) = \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_{\mathcal{B}} [G_{\mu,\alpha+i\epsilon} - G_{\mu,\alpha-i\epsilon}]\, \mathrm{d}\alpha$$

where $G_{\mu,\alpha}$ is the resolvent

$$G_{\mu,\alpha} \equiv (N_\mu - \alpha)^{-1}$$

and $\mathcal{B}$ is a (Borel) subset of $\mathbb{R}$. The kernel $G_{\mu,\alpha}(u, v)$ of $G_{\mu,\alpha}$ is determined by demanding that for each $v \in \mathbb{R}_+$, as a function of $u \in \mathbb{R}_+$, it is in the domain of N_μ and that it satisfies the equation

$$(N(u) - \alpha)\, G_{\mu,\alpha}(u, v) = \delta(u - v)\,. \tag{67}$$

To calculate $G_{\mu,\alpha}(u, v)$ we note that for $u \neq v$ we simply have $(N(u) - \alpha)G_{\mu,\alpha}(u, v) = 0$. Viewed as a function of u, $G_{\mu,\alpha}(u, v)$ verifies at zero the boundary condition (42) and should go to zero at infinity. A solution of $(N_\mu - \alpha)\phi(u) = 0$ satisfying the correct boundary condition at zero is

$$\phi_{\mu,\alpha}(u) = u^{\frac{1}{2}} \left[I_{-b}(\sqrt{-\alpha}\, u) + \mu(-\alpha)^{1-b} I_b(\sqrt{-\alpha}\, u) \right], \qquad \alpha \notin \mathbb{R}_+\,. \tag{68}$$

Here $I_{\pm b}$ is the modified Bessel function of parameter $\pm b$. The square-root is taken with its principal definition. Similarly, the solution of $(N - \alpha)\psi(u) = 0$ decaying at infinity is

$$\psi_\alpha(u) = u^{1/2} K_b(\sqrt{-\alpha} u)\,, \qquad \alpha \notin \mathbb{R}_+ \tag{69}$$

where K_b is the modified Bessel function of the second kind of parameter b. With the correct matching at $u = v$ to assure that (67) is satisfied, we obtain

$$G_{\mu,\alpha}(u, v) = \begin{cases} \dfrac{\phi_{\mu,\alpha}(u)\psi_\alpha(v)}{w_{\mu,\alpha}} & \text{if} \quad u \leqslant v\,, \\ \dfrac{\psi_\alpha(u)\phi_{\mu,\alpha}(v)}{w_{\mu,\alpha}} & \text{if} \quad u \geqslant v \end{cases} \tag{70}$$

where $w_{\mu,\alpha}$ is the Wronskian of $\phi_{\mu,\alpha}$ and ψ_α, i.e.

$$w_{\mu,\alpha} \equiv \phi'_{\mu,\alpha}(z)\psi_\alpha(z) - \phi_{\mu,\alpha}(z)\psi'_\alpha(z)$$

which is independent of z. To evaluate the Wronskian, we may use the asymptotic expansions for $z \to +\infty$

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}(1 + O(\frac{1}{z}))\,, \qquad K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z}(1 + O(\frac{1}{z}))\,,$$

see Eq. 8.451.6,7 of ref. 12. The result is

$$w_{\mu,\alpha} = 1 + \mu(-\alpha)^{1-b}\,.$$

Next we have to calculate the discontinuity in $G_{\mu,\alpha}(u,v)$ along the cut for $\alpha \in \mathbb{R}_+$. It will be convenient to write $-(\alpha \pm i0) = e^{\mp\pi i}\alpha$. This will automatically give the correct determination of every function. For $u < v$

$$\begin{aligned} G_{\mu,\alpha\pm i0}(u,v) = {} & \frac{1}{1+\mu\alpha^{1-b}e^{\mp\pi i(1-b)}} \\ & u^{1/2}\Big[I_{-b}(e^{\mp\frac{\pi i}{2}}\sqrt{\alpha}u) + \mu e^{\mp\pi i(1-b)}\alpha^{1-b} I_b(e^{\mp\frac{\pi i}{2}}\sqrt{\alpha}u)\Big] \\ & v^{1/2} K_b(e^{\mp\frac{\pi i}{2}}\sqrt{\alpha}v)\,. \end{aligned}$$

Now we employ the general formula

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi} \qquad \text{for} \quad \nu \notin \mathbb{Z}\,,$$

see Eq. 8.485 of ref. 12, to get

$$\begin{aligned} G_{\mu,\alpha\pm i0}(u,v) = {} & \frac{1}{1+\mu\alpha^{1-b}e^{\mp\pi i(1-b)}} \\ & u^{1/2}\Big[I_{-b}(e^{\mp\frac{\pi i}{2}}\sqrt{\alpha}u) + \mu e^{\mp\pi i(1-b)}\alpha^{1-b} I_b(e^{\mp\frac{\pi i}{2}}\sqrt{\alpha}u)\Big] \\ & v^{1/2}\frac{\pi}{2\sin b\pi}\Big[I_{-b}(e^{\mp\frac{\pi i}{2}}\sqrt{\alpha}v) - I_b(e^{\mp\frac{\pi i}{2}}\sqrt{\alpha}v)\Big]. \end{aligned}$$

The relations

$$I_\nu(e^{\frac{\pi i}{2}}z) = e^{\frac{\pi i}{2}\nu}J_\nu(z)\,, \qquad I_\nu(e^{-\frac{\pi i}{2}}z) = e^{-\frac{\pi i}{2}\nu}J_\nu(z)$$

following from Eqs.,8.406.1 and 8.476.1 of ref. 12 permit to move the determination from the function argument to a multiplicative coefficient. The final expression is

$$\begin{aligned} G_{\mu,\alpha\pm i0}(u,v) \;=\; & \frac{\pi}{2\sin b\pi}\,\frac{1}{1+\mu\alpha^{1-b}e^{\mp\pi i(1-b)}} \\ & \cdot\; u^{1/2}\Big[e^{\mp\frac{\pi i}{2}b}J_{-b}(\sqrt{\alpha}u) + \mu e^{\mp\pi i(1-b)}\alpha^{1-b}e^{\mp\frac{\pi i}{2}b}J_b(\sqrt{\alpha}u)\Big] \\ & \cdot\; v^{1/2}\Big[e^{\mp\frac{\pi i}{2}b}J_{-b}(\sqrt{\alpha}v) - e^{\mp\frac{\pi i}{2}b}J_b(\sqrt{\alpha}v)\Big]. \end{aligned}$$

It is now easy to calculate the discontinuity $G_{\mu,\alpha+i0}-G_{\mu,\alpha-i0}$ and to get for $\alpha=E\in\mathbb{R}_+$

$$dE_\mu(u,v) \;=\; \frac{\mathrm{dE}}{2(1-2\mu E^{1-b}\cos b\pi+\mu^2E^{2(1-b)})}\,\varphi_{\mu,E}(u)\,\varphi_{\mu,E}(v) \quad (71)$$

where $\varphi_{\mu,E}(u)$ given by (44) is the (generalized) eigenfunction of N_μ associated with the eigenvalue E. This corresponds to the scalar spectral measure (45).

APPENDIX E. CONSERVATION OF PROBABILITY

We shall show here that the transition measures $P^t_{\tilde\lambda}(r;\mathrm{dr}')$ of the sticky process given by (41) are normalized, i.e. that $\int P^t_{\tilde\lambda}(r,\mathrm{dr}')=1$. This is clearly the case for $t=0$ since $P^0_{\tilde\lambda}(r;\mathrm{dr}')=\delta(\mathrm{r}-\mathrm{r}')\,\mathrm{dr}'$. But

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int e^{tM_{\tilde\lambda}}(r,r')\,\mathrm{dr}' &= \int M^\dagger(r')\,e^{tM_{\tilde\lambda}}(r,r')\,\mathrm{dr}'\\ &= -F(r')\left[e^{tM_{\tilde\lambda}}(r,r')\,m(r')^{-1}\right]_{r'=0}\end{aligned}$$

where $M^\dagger=m(r)M\circ m(r)^{-1}=\partial_r\circ F\circ m(r)^{-1}$ with $m(r)=r^{a_\xi-\xi}$ and $F=\tilde D_0 r^{a_\xi}\partial_r$ is the formal adjoint of M w.r.t. the $L^2(\mathrm{d}r)$ scalar product. The expression on the right hand side has the interpretation of the flux of the probability current through $r'=0$. It should be balanced by the rate of change of probability to stay at $r'=0$. Using the relations

$$e^{tM_{\tilde\lambda}}(r,r')\,m(r')^{-1}=e^{tM_{\tilde\lambda}}(r',r)\,m(r)^{-1}$$

and the fact that $e^{tM_{\tilde\lambda}}(r,r')$ satisfies as a function of r the boundary condition (37), we infer that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int e^{tM_{\tilde\lambda}}(r,r')\,\mathrm{dr}' &= -\tilde\lambda\,M(r')\,e^{tM_{\tilde\lambda}}(r',r)\,m(r)^{-1}\Big|_{r'=0}\\ &= -\tilde\lambda\,M(r')\Big(e^{tM_{\tilde\lambda}}(r,r')\,m(r')^{-1}\Big)\Big|_{r'=0}\\ &= -\tilde\lambda\,m(r')^{-1}M^\dagger(r')\,e^{tM_{\tilde\lambda}}(r,r')\Big|_{r'=0}\\ &= -\tilde\lambda\,\frac{\mathrm{d}}{\mathrm{dt}}\,m(r')^{-1}\,e^{tM_{\tilde\lambda}}(r,r')\Big|_{r'=0}.\end{aligned}$$

It follows then from (41) that the time derivative of $\int P^t_{\tilde\lambda}(r;\mathrm{dr}')$ vanishes so that the normalization of $P^t_{\tilde\lambda}(r;\mathrm{dr}')$ does not change in time.

APPENDIX F. ESTIMATES OF FUNCTIONS f_1 and f_2

We present bounds showing in which sense the approximations given for functions f_1 and f_2/f_1 in Table II are correct.

The functions $f_{1,2}(r; l_\nu, l_\kappa)$ were defined by (14) and (15) with κ given by (28). Let us examine their behavior for $\kappa = 0$ setting $f_{1,2}(r; l_\nu) \equiv f_{1,2}(r; l_\nu, 0)$. Let us first show that $f_{1,2}(r; l_\nu)$ are positive for $r > 0$. Note that (6) implies that

$$d_{ij}(\vec{r}; l_\nu) = D_0 \int \frac{2\sin^2 \frac{\vec{k}.\vec{r}}{2}}{|\vec{k}|^{d+\xi}} P_{ij}(\vec{k}, \wp) f(l_\nu |\vec{k}|) \frac{\mathrm{d}\vec{\mathrm{k}}}{(2\pi)^d} = \tilde{D}_0 \int T_{ij}(\vec{k})\, \mathrm{d}\vec{\mathrm{k}}$$

where $T(\vec{k})$ is a positive ($\vec{r}$-dependent) matrix. On the other hand, (14) and (15) may be written as

$$f_1(r) = \int T_{ij}(\vec{k}) \frac{r_i r_j}{r^2} \, \mathrm{d}\vec{\mathrm{k}}, \qquad \mathrm{f_2(r) = r^{-1}} \int \mathrm{T_{ij}(\vec{k})} \left(\delta_{\mathrm{ij}} - \frac{\mathrm{r_i r_j}}{\mathrm{r^2}}\right) \mathrm{d}\vec{\mathrm{k}}$$

for any vector $\vec{r}$ such that $|\vec{r}| = r$. Positivity of T implies

$$T_{ij}(\vec{k})\, r_i r_j \geqslant 0, \qquad T_{ij}(\vec{k})(\delta_{ij} - \tfrac{r_i r_j}{r^2}) \geqslant 0$$

and for $r > 0$ it is not hard to see that strict equalities hold except for a set of $\vec{k}$ of measure zero. Hence positivity of $f_{1,2}(r; l_\nu)$.

In order to study the behavior of $f_{1,2}(r; l_\nu)$ for small and large r it will be convenient to express them in terms of the function

$$g(r; l_\nu) = \int \frac{1 - e^{i\vec{k}\cdot\vec{r}}}{|\vec{k}|^{d+\xi}} f(l_\nu |\vec{k}|) \frac{\mathrm{d}\vec{\mathrm{k}}}{(2\pi)^d}.$$

Note that g is a smooth even function (i.e. all its derivatives of odd order vanish at 0) of $r \geqslant 0$. By scaling,

$$g(r; l_\nu) = l_\nu^\xi \, g(\tfrac{r}{l_\nu}; 1).$$

Around zero

$$g(\rho; 1) = C_0 \rho^2 + O(\rho^4) \qquad \text{with} \qquad C_0 = \int \frac{f(|\vec{k}|)}{|\vec{k}|^{d+\xi-2}} \frac{\mathrm{d}\vec{\mathrm{k}}}{(2\pi)^d}. \tag{72}$$

Rewriting

$$g(\rho;1)=\rho^{\xi}\int\frac{1-e^{i\vec{k}\cdot\vec{\rho}/\rho}}{|\vec{k}|^{d+\xi}}\,f(|\vec{k}|/\rho)\,\frac{\mathrm{d}\vec{k}}{(2\pi)^{d}}\,,$$

we infer that around infinity

$$g(\rho;1)=C_{\infty}\rho^{\xi}\,+\,O(\rho^{\xi-2}) \tag{73}$$

with

$$C_{\infty}=\int\frac{1-e^{i\vec{k}\cdot\vec{\rho}/\rho}}{|\vec{k}|^{d+\xi}}\,\frac{\mathrm{d}\vec{k}}{(2\pi)^{d}}=\frac{\Gamma(\frac{2-\xi}{2})}{2^{d+\xi-2}\pi^{\frac{d}{2}}\xi\,\Gamma(\frac{d+\xi}{2})}\,.$$

The spatial covariance $\vec{\vec{d}}(\vec{r};l_{\nu})$ given by (6) may be expressed in terms of the function g:

$$\begin{aligned}d_{ij}(\vec{r};l_{\nu})=D_{0}\Big[\Big(\frac{1-\wp}{d-1}\delta_{ij}+\frac{\wp d-1}{d-1}\frac{r_{i}r_{j}}{r^{2}}\Big)\,g(r;l_{\nu})\\+\frac{\wp d-1}{d-1}\Big(\delta_{ij}-d\frac{r_{i}r_{j}}{r^{2}}\Big)\,r^{-d}\int_{0}^{r}(r')^{\,d-1}g(r';l_{\nu})\,\mathrm{d}r'\Big].\end{aligned} \tag{74}$$

Indeed, by rotational covariance, d_{ij} has to be a combination of δ_{ij} and $\frac{r_{i}r_{j}}{r^{2}}$. To show that the coefficient functions are correctly represented above, it is enough to note that d_{ii} and $\partial_{i}d_{ij}$ are, in virtue of (6), expressible in terms of g and of its r-derivative g' as $d_{ii}=D_{0}g$ and $\partial_{i}d_{ij}=D_{0}\wp\frac{r_{j}}{r}g'$ and that the same relations may be recovered from (74). Now using definitions (14) and (15) we obtain:

$$\frac{\tilde{D}_{0}}{D_{0}}\,f_{1}(r;l_{\nu})=\frac{1-\wp}{d-1}\,g(r;l_{\nu})+\frac{\wp d-1}{d-1}\left(g(r;l_{\nu})-\frac{d-1}{r^{d}}\int_{0}^{r}r'^{\,d-1}\,g(r';l_{\nu})\,\mathrm{d}r'\right), \tag{75}$$

$$\frac{\tilde{D}_{0}}{D_{0}}\,f_{2}(r;l_{\nu})=\frac{1-\wp}{r}\,g(r;l_{\nu})+\frac{\wp d-1}{r^{d}}\int_{0}^{r}r'^{\,d-1}\,g(r';l_{\nu})\,\mathrm{d}r'\,. \tag{76}$$

Functions $f_{1,2}(r;l_{\nu})$ inherit from $g(r;l_{\nu})$ the scaling property (55). Relations (75), (76) and expansions (72) and (73) permit now to write the decompositions (56) to (59) with $\tilde{C}_{0}=\frac{1+2\wp}{d+2}C_{0}D_{0}/\tilde{D}_{0}$ and similar coefficient in (58) and (59) fixed to 1 by setting $\tilde{D}_{0}=\frac{1+\wp\xi}{d+\xi}C_{\infty}D_{0}$. Moreover the positivity of functions $f_{1,2}(r;1)$ and their smoothness imply that there exists some constant $B>0$ such that for all $0\le\rho\le 1$

$$\frac{1}{B} < f_i^{\clubsuit}(\rho) < B \qquad i=1,2, \tag{77}$$

$$-B < f_i^{\spadesuit}(\rho) < B \qquad i=1,2 \tag{78}$$

and for all $1\le\rho<\infty$,

$$\frac{1}{B} < f_i^{\diamondsuit}(\rho) < B \qquad i=1,2, \tag{79}$$

$$-B < f_i^{\heartsuit}(\rho) < B \qquad i=1,2. \tag{80}$$

Let us come back to functions $f_{1,2}(r;l_\nu,l_\kappa)$ with $0<l_\kappa,l_\nu<1$. To obtain bounds for f_1, we write for $l_\nu<r$

$$f_1(r;l_\nu,l_\kappa)=r^\xi f_1^{\diamondsuit}(\frac{r}{l_\nu})+2l_\nu^{\xi-2}l_\kappa^2$$

and infer that

$$r^\xi\frac{1}{B} < f_1(r;l_\nu,l_\kappa) < r^\xi(B+2). \tag{81}$$

Similarly, the decomposition

$$f_1(r;l_\nu,l_\kappa)=\tilde{C}_0\, l_\nu^{\xi-2}r^2 f_1^{\clubsuit}(\frac{r}{l_\nu})+2l_\nu^{\xi-2}l_\kappa^2$$

gives for $l_\kappa<r<l_\nu$ the bounds

$$l_\nu^{\xi-2}r^2\frac{\tilde{C}_0}{B} < f_1(r;l_\nu,l_\kappa) < l_\nu^{\xi-2}r^2(\tilde{C}_0B+2)$$

and for $r<l_\kappa$,

$$2l_\nu^{\xi-2}l_\kappa^2 < f_1(r;l_\nu,l_\kappa) < l_\nu^{\xi-2}l_\kappa^2(\tilde{C}_0B+2). \tag{82}$$

This shows that $f_1(r;l_\nu,l_\kappa)$ is bounded above and below by its approximate version of Table II multiplied or divided by a constant.

Function f_2 may be estimated similarly resulting in $O(1)$ bounds for rf_2/f_1. We need however (in particular in Appendix G) the following more precise estimates for that ratio:

$$\text{for } l_\nu < r : \qquad \left|\frac{f_2(r;l_\nu,l_\kappa)}{f_1(r;l_\nu,l_\kappa)} - \frac{a_\xi}{r}\right| < C\frac{l_\nu^\xi}{r^{1+\xi}} \tag{83}$$

$$\text{for } l_\kappa < r < l_\nu : \qquad \left|\frac{f_2(r;l_\nu,l_\kappa)}{f_1(r;l_\nu,l_\kappa)} - \frac{a_2}{r}\right| < C\left[\frac{r}{l_\nu^2} + \frac{l_\kappa^2}{r^3}\right] \tag{84}$$

$$\text{for } r < l_\kappa : \qquad \left|\frac{f_2(r;l_\nu,l_\kappa)}{f_1(r;l_\nu,l_\kappa)} - \frac{d-1}{r}\right| < C\frac{r}{l_\kappa^2} \tag{85}$$

for some constant C which may be chosen independent of r, l_ν, l_κ. To establish (83) we use the decompositions (58) and (59) to write for $l_\nu < r$

$$\begin{aligned}
&\left|\frac{f_2(r;l_\nu,l_\kappa)}{f_1(r;l_\nu,l_\kappa)} - \frac{a_\xi}{r}\right| \\
&\quad = \left|\frac{l_\nu^{\xi-1} f_2(\frac{r}{l_\nu};1) + 2\frac{d-1}{r} l_\nu^{\xi-2} l_\kappa^2 - \frac{a_\xi}{r}\left[l_\nu^\xi f_1(\frac{r}{l_\nu};1) + 2l_\nu^{\xi-2} l_\kappa^2\right]}{l_\nu^\xi f_1(\frac{r}{l_\nu};1) + 2l_\nu^{\xi-2} l_\kappa^2}\right| \\
&\quad = \left|\frac{l_\nu^2 r^{\xi-3}\left[f_2^\heartsuit(\frac{r}{l_\nu}) - a_\xi f_1^\heartsuit(\frac{r}{l_\nu})\right] + 2\frac{d-1-a_\xi}{r} l_\nu^{\xi-2} l_\kappa^2}{r^\xi f_1^\diamondsuit(\frac{r}{l_\nu}) + 2l_\nu^{\xi-2} l_\kappa^2}\right| \\
&\quad < \frac{B[1+|a_\xi|] + 2|d-1-a_\xi|}{\frac{1}{B}}\,\frac{l_\nu^\xi}{r^{1+\xi}}
\end{aligned}$$

where the inequality follows from the bounds (79) and (80). In the same manner for $l_\kappa < r < l_\nu$, using the decompositions (56), (57) and the bounds (77) and (78), we obtain

$$\begin{aligned}
\left|\frac{f_2(r;l_\nu,l_\kappa)}{f_1(r;l_\nu,l_\kappa)} - \frac{a_2}{r}\right| &= \left|\frac{l_\nu^{\xi-4} r^3\left[f_2^\spadesuit(\frac{r}{l_\nu}) - a_2 f_1^\spadesuit(\frac{r}{l_\nu})\right] + 2\frac{d-1-a_2}{r} l_\nu^{\xi-2} l_\kappa^2}{\tilde{C}_0 l_\nu^{\xi-2} r^2 f_1^\clubsuit(\frac{r}{l_\nu}) + 2l_\nu^{\xi-2} l_\kappa^2}\right| \\
&< \left[\frac{r}{l_\nu^2} + \frac{l_\kappa^2}{r^3}\right]\frac{B[1+|a_2|] + 2|d-1-a_2|}{\frac{\tilde{C}_0}{B}}.
\end{aligned}$$

Finally for $r < l_\kappa$,

$$\left|\frac{f_2(r;l_\nu,l_\kappa)}{f_1(r;l_\nu,l_\kappa)} - \frac{d-1}{r}\right|$$

$$= \left| \frac{\tilde{C}_0(a_2-d+1)l_\nu^{\xi-2}\,r + l_\nu^{\xi-4} r^3 \left[f_2^{\spadesuit}(\tfrac{r}{l_\nu}) - (d-1) f_1^{\spadesuit}(\tfrac{r}{l_\nu}) \right]}{\tilde{C}_0 l_\nu^{\xi-2} r^2 f_1^{\clubsuit}(\tfrac{r}{l_\nu}) + 2 l_\nu^{\xi-2} l_\kappa^2} \right|$$
$$< \frac{r}{l_\kappa^2}\,\frac{\tilde{C}_0|d-1-a_2| + B[1+|d-1|]}{2}\,.$$

Let us end this Appendix by listing the scaling forms of the tensor $\vec{\vec{d}}$ and of functions $f_{1,2}$. Since $g(r;0) = C_\infty r^\xi$, we infer from (74) that

$$d_{ij}(\vec{r},0) = \frac{\tilde{D}_0}{d-1}\Big[\Big(\frac{d+\xi}{1+\xi\wp} - 1\Big)\delta_{ij} + \frac{\xi(\wp d - 1)}{1+\xi\wp}\,\frac{r_i r_j}{r^2}\Big] r^\xi$$

and that $f_1(r;0) = r^\xi$ and $f_2(r;0) = a_\xi\, r^{\xi-1}$, in agreement with the scaling form (17) of the generator M.

APPENDIX G. EXACT SPEED MEASURE AT ZERO

Here we prove the convergence (60), establishing the exact value of the mass at zero of the limiting speed measure. Using (23), we have

$$\int_0^{l_\nu} m(r';l_\nu,l_\kappa)\,\chi(r')\,\mathrm{d}r' = \int_0^{l_\nu} \frac{\chi(r')}{f_1(r';l_\nu,l_\kappa)}\,\exp\Big(-\int_{r'}^{r_0} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r''\Big)\,\mathrm{d}r'\,.$$

The integral in the exponential may be split into four terms:

$$\int_{r'}^{r_0} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r'' = \int_{r'}^{l_\kappa} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r''$$
$$+ \int_{l_\kappa}^{\sqrt{l_\kappa l_\nu}} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r'' + \int_{\sqrt{l_\kappa l_\nu}}^{l_\nu} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r'' + \int_{l_\nu}^{r_0} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r''\,.$$

We begin by calculating the last three terms as they do not depend on r'. The last one gives

$$\int_{l_\nu}^{r_0} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r''$$
$$= a_\xi \ln\frac{r_0}{l_\nu} + \int_1^{\frac{r_0}{l_\nu}} \frac{\rho^{\xi-3}\Big[f_2^{\heartsuit}(\rho) - a_\xi f_1^{\heartsuit}(\rho)\Big] + 2\frac{d-1-a_\xi}{\rho}\Big(\frac{l_\kappa}{l_\nu}\Big)^2}{\rho^\xi f_1^{\diamondsuit}(\rho) + 2\Big(\frac{l_\kappa}{l_\nu}\Big)^2}\,\mathrm{d}\rho$$

and we know from (83) that the integrand on the right is dominated by $C\rho^{-(1+\xi)}$ (as soon as $l_\nu < 1$), so we may use the Dominated Convergence Theorem to conclude that

$$\lim_{l_\nu, \frac{l_\kappa}{l_\nu}\to 0}\left(\int_{l_\nu}^{r_0} \frac{f_2(r''; l_\nu, l_\kappa)}{f_1(r''; l_\nu, l_\kappa)}\, \mathrm{d}r'' - \mathrm{a}_\xi \ln \frac{\mathrm{r}_0}{\mathrm{l}_\nu}\right) = \int_1^\infty \frac{f_2^\heartsuit(\rho) - a_\xi f_1^\heartsuit(\rho)}{\rho^3 f_1^\diamondsuit(\rho)}\, \mathrm{d}\rho\,.$$

As for the other two,

$$\int_{l_\kappa}^{\sqrt{l_\kappa l_\nu}} \frac{f_2(r''; l_\nu, l_\kappa)}{f_1(r''; l_\nu, l_\kappa)}\, \mathrm{d}r''$$
$$= a_2 \ln\sqrt{\frac{l_\nu}{l_\kappa}} + \int_1^{\sqrt{l_\nu/l_\kappa}} \frac{\left(\frac{l_\kappa}{l_\nu}\right)^2 \rho^3 \left[f_2^\spadesuit(\frac{l_\kappa}{l_\nu}\rho) - a_2 f_1^\spadesuit(\frac{l_\kappa}{l_\nu}\rho)\right] + 2\frac{d-1-a_2}{\rho}}{\tilde{C}_0 \rho^2 f_1^\clubsuit(\frac{l_\kappa}{l_\nu}\rho) + 2}\, \mathrm{d}\rho$$

with the integrand on the right dominated by $C[(\frac{l_\kappa}{l_\nu})^2\rho + \rho^{-3}] < 2C\rho^{-3}$, see (84), so we may conclude that

$$\lim_{l_\nu, \frac{l_\kappa}{l_\nu}\to 0}\left(\int_{l_\kappa}^{\sqrt{l_\kappa l_\nu}} \frac{f_2(r''; l_\nu, l_\kappa)}{f_1(r''; l_\nu, l_\kappa)}\, \mathrm{d}r'' - \mathrm{a}_2 \ln\sqrt{\frac{\mathrm{l}_\nu}{\mathrm{l}_\kappa}}\right)$$
$$= \int_1^\infty \frac{2(d-1-a_2)}{\rho\,(\tilde{C}_0\rho^2 + 2)}\, \mathrm{d}\rho = \frac{\mathrm{d}-1-\mathrm{a}_2}{2} \ln \frac{\tilde{\mathrm{C}}_0 + 2}{\tilde{\mathrm{C}}_0}\,.$$

Similarly,

$$\int_{\sqrt{l_\kappa l_\nu}}^{l_\nu} \frac{f_2(r''; l_\nu, l_\kappa)}{f_1(r''; l_\nu, l_\kappa)}\, \mathrm{d}r''$$
$$= a_2 \ln\sqrt{\frac{l_\nu}{l_\kappa}} + \int_{\sqrt{l_\kappa/l_\nu}}^1 \frac{\rho^3\left[f_2^\spadesuit(\rho) - a_2 f_1^\spadesuit(\rho)\right] + 2\frac{d-1-a_2}{\rho}\left(\frac{l_\kappa}{l_\nu}\right)^2}{\tilde{C}_0\rho^2 f_1^\clubsuit(\rho) + 2\left(\frac{l_\kappa}{l_\nu}\right)^2}\, \mathrm{d}\rho$$

where the integrand on the right is bounded by $C[\rho + (\frac{l_\kappa}{l_\nu})^2\rho^{-3}] < 2C\rho$, see (84). Hence

$$\lim_{l_\nu, \frac{l_\kappa}{l_\nu}\to 0}\left(\int_{\sqrt{l_\kappa l_\nu}}^{l_\nu} \frac{f_2(r''; l_\nu, l_\kappa)}{f_1(r''; l_\nu, l_\kappa)}\, \mathrm{d}r'' - \mathrm{a}_2 \ln\sqrt{\frac{\mathrm{l}_\nu}{\mathrm{l}_\kappa}}\right) = \int_0^1 \frac{\rho\left[f_2^\spadesuit(\rho) - a_2 f_1^\spadesuit(\rho)\right]}{\tilde{C}_0 f_1^\clubsuit(\rho)}\, \mathrm{d}\rho\,.$$

What remains to be evaluated is

$$\int_0^{l_\nu} \frac{\chi(r')}{f_1(r';l_\nu,l_\kappa)} \exp\left(-\int_{r'}^{l_\kappa} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r''\right) \mathrm{d}r'$$
$$= \frac{1}{l_\nu^{\xi-2} l_\kappa} \int_0^{l_\nu/l_\kappa} \frac{\chi(l_\kappa\rho)}{\tilde{C}_0\rho^2 f_1^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho)+2}$$
$$\cdot \exp\left[-\int_\rho^1 \frac{\tilde{C}_0 a_2 \rho' f_2^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho')+2\frac{d-1}{\rho'}}{\tilde{C}_0\rho'^2 f_1^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho')+2}\,\mathrm{d}\rho'\right] \mathrm{d}\rho\,.$$

Let us show that the Dominated Convergence Theorem applies once more. The following estimates are sufficient. For $0<\rho<1$,

$$\left|\frac{\chi(l_\kappa\rho)}{\tilde{C}_0\rho^2 f_1^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho)+1}\right| < \chi(0)\,, \qquad \left|\frac{\tilde{C}_0 a_2\rho f_2^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho)+\frac{d-1}{\rho}}{\tilde{C}_0\rho^2 f_1^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho)+1} - \frac{d-1}{\rho}\right| < C\rho\,.$$

The first inequality is a consequence of $\chi(r)<\chi(0)$ for any $r>0$, and of the positivity of $f_1^{\clubsuit}$. The second one is obtained by rewriting (85) using (56) and (57). For $1<\rho<l_\nu/l_\kappa$, we use

$$\left|\frac{\chi(l_\kappa\rho)}{\tilde{C}_0\rho^2 f_1^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho)+1}\right| < \frac{\chi(0)}{\frac{\tilde{C}_0}{B}\rho^2}$$
$$\left|\frac{\tilde{C}_0 a_2\rho f_2^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho)+\frac{d-1}{\rho}}{\tilde{C}_0\rho^2 f_1^{\clubsuit}(\frac{l_\kappa}{l_\nu}\rho)+1} - \frac{a_2}{\rho}\right| < C\left[\left(\frac{l_\kappa}{l_\nu}\right)^2\rho + \frac{1}{\rho^3}\right].$$

Here, the first inequality follows from (77) and the second one is obtained by rewriting (84) with the use of (56) and (57). We then have

$$\lim_{l_\nu,\frac{l_\kappa}{l_\nu}\to 0} l_\nu^{\xi-2} l_\kappa \int_0^{l_\nu} \frac{\chi(r')}{f_1(r';l_\nu,l_\kappa)} \exp\left(-\int_{r'}^{l_\kappa} \frac{f_2(r'';l_\nu,l_\kappa)}{f_1(r'';l_\nu,l_\kappa)}\,\mathrm{d}r''\right) \mathrm{d}r'$$
$$= \int_0^\infty \frac{\chi(0)}{\tilde{C}_0\rho^2+2} \exp\left[-\int_\rho^1 \frac{\tilde{C}_0 a_2\rho' + 2\frac{d-1}{\rho'}}{\tilde{C}_0\rho'^2+1}\,\mathrm{d}\rho'\right] \mathrm{d}\rho$$
$$= 2^{\frac{a_2-3}{2}} (\tilde{C}_0+2)^{\frac{d-1-a_2}{2}} \tilde{C}_0^{-\frac{d}{2}} B(\tfrac{d}{2},\tfrac{1-a_2}{2})\,\chi(0)$$

where $B(x,y)$ is the Euler Beta function. Gathering all terms, in particular the powers of l_ν and l_κ that combine to $l_\nu^{(a_\xi+1-\xi)+(1-a_2)} l_\kappa^{a_2-1}$, we obtain the result (60) with Y given by (61).

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